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What we call objective reality is, in the last analysis, what is common to many thinking beings and could be common to all; this common part . . . can only be the harmony expressed by mathematical laws.

—H. Poincaré

PREFACE

This book with additional exercises and formulas for reference is an abridgment and re-arrangement of the text of *Applied Calculus*, published in 1919.

It is issued to supply the demand for a shorter course — to provide without omissions a text book better adapted to the time available for the subject in the usual college course.

Comprised in pages about one hundred less in number than those of the original edition, it is believed that it may without omissions be a suitable text for an academic year.

Designed as a first course in the calculus, it is sufficiently advanced for use in colleges and technical schools, presenting some features not usually found in a first course. While designed too to precede a course in technical mechanics, the illustrations involve the fundamentals of that subject, and it may prove a helpful introduction to that study. The applications and concrete illustrations form a considerable part of the subject matter, but they have been chosen to enforce the essential principles of the calculus, and while they have been taken from geometry and mechanics as well as from general physics and astronomy, for their understanding and for the solution of the related problems, no special knowledge is required of the student. Although the text has been prepared primarily for technical students and for engineers, the effort has been to present in their unity the essential principles of Differential and Integral Calculus, and thus to provide a text book suited for classes in non-technical colleges also.

In the order of the treatment no rigid separation has been made between the theory and the applications, but

the latter are interspersed through the text,— not reserved for the end where, generally from lack of time, they get scant attention. The intention has been to produce a text in line with modern ideas — to meet the recent demand for the presentation of mechanical principles and their application in engineering practice at an early period of the student's course.

The essential integration processes are given, but the space, usually taken up by algebraic reductions and transformations, is saved for more profitable discussions. To conserve their unity the original plan of treating the two parts of the Calculus separately has been adhered to. The method of rates and the method of limits, involving the use of infinitesimals, are both employed, and the advantages of each method made apparent.

Attention is invited to the geometric deduction of the derivative as a limit, the fundamental conception of the Differential Calculus. This deduction may be admitted as valid even by those writers who with emphasis tell the reader to regard as only a symbol what undoubtedly appears to be more, and what they later are constrained to explain may be just that which it appears to be.

Attention is called to the easy way in which Taylor's Theorem results by induction and integration,— the reason for having the Theorem in Part II rather than in Part I, where it is usually found by a laborious deduction.

What has been omitted from the original edition does not break the continuity of the treatment of the principles of the calculus. The subjects omitted are: *Moment of Inertia, Center of Gravity, Infinite Series, Differential Equations* and *Central Forces* — only a few special applications being omitted. The title of this edition implies that it is not as complete as the original, but the opinions of instructors of the calculus, expressed to the publishers, have led to the preparation and publication of this volume. The origi-

nal book is more adapted to the needs of practicing engineers, and copies are still available.

In addition to the references in the text and in foot notes, acknowledgment is here made of suggestive ideas found in the books on the Calculus by *Gibson*, by *Taylor*, by *Davis*, and by *Townsend* and *Goodenough;* also in *Hedrick's* paper on the *Calculus without Symbols*.

The introduction to this book ends with a reference to the discoverer of the Calculus. It is deemed not unfitting that the book should close with *Newton's Verification*.

ROBERT GIBBES THOMAS.

COLUMBIA, S. C.,
Sept. 15, 1924.

PREFACE TO APPLIED CALCULUS, 1919

This book as a first course in the Calculus is not designed to be a complete exposition of the Calculus in either its principles or its applications. It is an effort to make clear the basic principles and to show that fundamental ideas are involved in familiar problems. While formulas and algebraic methods are necessary aids to concise and formal presentation, they are not essential to the expression of the principles and underlying ideas of the Calculus. These can be expressed in plain language without the use of symbols — one writer challenging the citing of a single instance where it cannot be done.

The practice is common, at least with "thoughtless thinkers," of blindly using formulas without any true conception of the ideas for which they are but the symbolic expression. The formulas of the Calculus are an invaluable aid in economy of thought, but their effective use is dependent upon an adequate knowledge of their derivation. The object of this book is to set forth the methods of the Calculus in such a way as to lead to a working and fruitful knowledge of its elements, to exhibit something of its power, and to induce its use as an efficient tool. No claim is made for absolute rigor in all the deductions, but confidence is invited in the soundness of the reasoning employed and in the logical conclusions obtained.

There are students, and engineers also, who when constrained to use the Calculus look upon it as a necessary evil This attitude is without doubt due to their minds having never had a firm grasp upon its principles nor a full realiza-

tion of the efficiency of its methods. A student while taking a course in the Calculus usually spends at least one-half his effort in reviewing previous mathematics. In fact, a course in the Calculus is held to involve an excellent review of geometry, algebra, and especially of trigonometry; hence, at the end of a term it is too much to expect of the average student that he have an adequate knowledge of the Calculus.

If a choice must be made between the ability to solve equations (including integration processes) and the far more rare ability to set up equations to represent established facts and laws, there can be little question as to which type of ability should be cultivated. The latter is of higher order and is likely to include the former. Engineers, physicists, inventors and men of science generally find it difficult to translate their observations into language which the pure mathematician can understand. In fact, such translation usually involves the writing of the equation: an undertaking beyond the capacity equally of the non-mathematical scientist and the pure mathematician. Integration of the equation, once set up, the mathematician will undertake; conceivably, so might a machine. Fruitful deductions and rules of practice result. The difficulty of realizing these results arises not from difficulties in moving about the symbols, but from inability on the part of nearly all persons to state facts in terms of symbols. It is as if no harmonist knew a melody and no melodist knew a note. This book aims to keep fact and symbol in close association, so that the student will never use the latter without being conscious of the former. It may then be expected that he will ultimately be able to visualize the symbolic expression when the fact is known.

Apart from the references in the text and in footnotes, acknowledgment is here made of the clarifying and logical ideas embodied in the books on the Calculus by *Gibson*, by *Taylor*, and by *Townsend* and *Goodenough;* also in *Hedrick's* paper on the *Calculus without Symbols.*

The introduction to this book ends with a reference to the discoverer of the Calculus. It is deemed not unfitting that the book should close with the *Central Forces of the Principia.*

ROBERT GIBBES THOMAS.

THE CITADEL, CHARLESTON, S. C.
FEBRUARY 1st, 1919.

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APPLIED CALCULUS.

INTRODUCTION.

THE Calculus treats of the rates of change of related variables. The factors of life are ever changing, acting and reacting upon each other. The quantities with which we have to deal in ordinary affairs are for the most part in a state of change. Hence the field in which the principles of the Calculus are directly involved is a wide one.

In observing the changes about us we note that they take place at various rates, and the determination of the rapidity of the change may be the controlling factor in many investigations. Whenever the rapidity of the change of anything is in question, the methods of the Calculus have appropriate application.

In the case of velocity or speed, there is rate of change of distance and time; in a thermometer we have rate of change of length and temperature, while in the barometer there is rate of change of height and density; in the slope of ground or grade of a road we have rate of change of vertical height and horizontal distance; and in the case of a curve, rate of change of ordinate and abscissa, or slope of the curve.

In the case of a body in variable motion, it becomes desirable to determine its velocity at some point of its path or at some instant of time, that is, the instantaneous velocity. This notion of rate of change at an instant is common even to untrained minds.

When one says of a train in variable motion, that it is now going at the rate of sixty miles an hour, one means that

at the instant considered the rate is such that, if it were maintained, the train would go sixty miles in an hour, that being the instantaneous velocity.

The method of the Calculus in getting the rate of change of a variable at any instant is in accordance with natural procedure: measure the amount of change in a short period of time, then the average rate of change during that period is the ratio of amount of change to length of period; the limit approached by this ratio, as the period of time is diminished towards zero as a limit, is the rate of change at the instant the period began.

In determining the greatest and least values of a variable quantity, they are found where the rate of change of the variable is zero. For instance, at the maximum and minimum temperature during a day, the rate of change of the temperature is zero. There is a difference, however, in that before the hottest moment the temperature was rising and then afterwards falling, while at the coldest moment the reverse was the case. In both cases the temperature's rate of change was momentarily zero. Here is to be seen the method of the Calculus as to maxima and mimina.

A distinguishing feature of the Calculus is that in addition to *real* sensible quantities it uses *ideal* hypothetical concepts, which are quantities that exist if certain conditions are maintained. The Calculus connects these two classes of quantities. Passing from the real to the ideal is *Differentiation*, from the ideal to the real is *Integration*. The advantage of introducing ideal quantities is that in many problems an expression for the ideal is readily formed and from this expression the real quantity is obtained by Integration. In other cases the real quantity being given, the problem is solved by the ideal quantity, obtained from the real by Differentiation.

The might of the invisible and intangible forces in Nature, predicated upon a concept (*the aether*), is generally recognized

in this day and generation. Therefore, it is not to be wondered at, that in dealing with material things and in seeking the inner law by which they act and react upon each other, we should call to our aid *ideal* concepts. The exclusive realist in his passion for *facts* is prone to overlook the fact that *ideas* are the first of facts.

It is acknowledged that science is useless unless it teaches us something about reality. Let it be acknowledged that the aim of science is not things themselves, but the relations between things, and the fruitfulness of the ideal quantities of the Calculus is recognized. The *differentials* employed, when properly defined, are not "ghosts of departed quantities," even if in some cases ideal in character. "*Airy,*" perhaps, but never "*nothing,*" they give to the creation of the mind "*a local habitation and a name.*"

While the ratio of some real quantities may never equal the ratio of the ideal quantities, nevertheless the former ratio may approach so closely the latter as a limit that the exact value of the ideal ratio can be discerned. So the differentiation of any real variable quantity is possible. On the other hand, the exact integration of every ideal quantity is not possible, and in some cases no corresponding real quantity exists.

In this respect there is an analogy with Involution and Evolution. Any number may be raised to a power; but the exact root of every number cannot be found, and no real root exists for some numbers.

Differential Calculus deals with the rates of change of continuous variables when the relation of the variables is known.

Integral Calculus is concerned with the inverse problem of finding the relation of the variables themselves when their relative rates are known.

While some problems to which the Calculus is applied may be solved by other methods, it often furnishes the simplest

solution; and there are cases in which the Calculus alone gives the solution. The Calculus is a tool for the efficient worker, and in the hands of skillful investigators the Calculus has proved to be a powerful instrument in bringing to light the truths of Nature.

In reference to the mighty intellect that conceived it, there is pardonable hyperbole in the lines of the Poet: —

> "*Nature and Nature's laws lay hid in night,*
> *God said, 'Let Newton be,' and all was light.*"

PART I.

DIFFERENTIAL CALCULUS.

CHAPTER I.

FUNCTIONS. DIFFERENTIALS. RATES.

1. Variables and Constants. — A *variable* is a quantity that changes in value. It is said to vary *continuously* when, in changing from one value to another, it takes successively each intermediate value once at least. If at any value it ceases to vary continuously, it is said to be discontinuous at that value.

A *constant* is a quantity whose value is fixed. If its value is always the same in every discussion, it is an *absolute* constant. If the fixed value may be different in different discussions, it is an *arbitrary* constant.

In the equation of the circle, $x^2 + y^2 = r^2$, x and y are variables that vary continuously from 0 to $\pm r$; while r is an arbitrary constant, as its value is fixed only for any one circle.

In the ordinary affairs of life we have to deal with continuous variables, such as time; the distance of an object in continuous motion from any point on its path; and with discontinuous variables, such as the amount of a sum of money at interest compounded periodically; the price of cotton; the cost of money orders, etc.

In nature we have constants, such as: the mass of a body, which is an absolute constant; the weight of a body, which is an arbitrary constant. as it is fixed according to latitude

and elevation; in mathematics, the ratio of the circumference of a circle to its diameter and the base of Naperian logarithms are absolute constants.

Variables are represented usually by the last letters of the alphabet; as x, y, z, or ρ, θ, ϕ, etc. The letter A, however, often represents a variable area.

Absolute constants are denoted by number symbols, and there are some absolute constants represented by letters, as π, e, for the ratio and base just mentioned, each transcendental but the most important in mathematics.

Arbitrary constants are represented usually by the first letters of the alphabet; as a, b, c, α, β, γ, etc. Particular values of variables are constants and are denoted by x_1, y_1, z_1, x_2, y_2, z_2, etc.

2. Functions. Dependent and Independent Variables. — When two variables are so related that the value of one of them depends upon the value of the other, the first is the *dependent variable* and is said to be a *function* of the second, and the second is the *independent variable*, which in connection with the function is usually called simply the *variable*, or sometimes the *argument*.

The area of a square is a function of the length of a side. The area or the circumference of a circle is a function of its radius. The square, or the square root, or the logarithm of a number, is a function of the number.

Any function of x is represented by $f(x)$, $F(x)$, $\phi(x)$, etc., and the symbol $f(x)$ denotes any expression involving x, whose value depends upon the value of x. In any discussion involving x, $f(a)$ means the value of $f(x)$ when x is replaced by a throughout the expression. In $y = f(x)$, x is the independent variable and y is the function or dependent variable. In the equation $x^2 + y^2 = r^2$, $y = \sqrt{r^2 - x^2}$ or $x = \sqrt{r^2 - y^2}$, so $y = f(x)$ or $x = f(y)$. If one variable is expressed directly in terms of another, the first is said to be an *explicit* function of the second. If the relation between

the two variables is given by an equation containing them but not solved for either, then either variable is said to be an *implicit* function of the other. So in $x^2 + y^2 = r^2$, y is an implicit function of x and x is an implicit function of y; but in $y = \sqrt{r^2 - x^2}$, y is an explicit function of x, and in $x = \sqrt{r^2 - y^2}$, x is an explicit function of y.

A variable may be a function of more than one variable; thus in $z^2 = x^2 + y^2$, or in $z = xy$, z is a function of x and y; the notation being $z = f(x, y)$. The area of a rectangle is a function of its base and altitude. The volume of a solid is a function of its three dimensions; so in $V = xyz$, $V = f(x, y, z)$.

3. Function — Continuous or Discontinuous. — A function as $f(x)$ is said to be continuous between $x = a$ and $x = b$, if when x varies continuously from a to b, $f(x)$ varies continuously from $f(a)$ to $f(b)$. In other words, $f(x)$ is continuous between $x = a$ and $x = b$ when the locus of $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ is an unbroken line, straight or curved.

A function is said to be discontinuous at any value when it ceases to vary continuously at that value, even though its variable may be continuous.

Some functions are continuous for all values of their variables; others are continuous only between certain limits. For example, $\sin\theta$ and $\cos\theta$ are continuous for all values of θ; $\tan\theta$ is continuous from $\theta = 0$ to $\theta = \pi/2$ and from $\theta = \pi/2$ to $\theta = \frac{3}{2}\pi$, but when θ passes through $\pi/2$ or $\frac{3}{2}\pi$, $\tan\theta$ changes from $+\infty$ to $-\infty$, hence $\tan\theta$ is discontinuous for $\theta = \pi/2$ or $\frac{3}{2}\pi$.

The Calculus treats of *continuous* variables and functions, or of variables and functions only between their limits of continuity.

4. Representation of Functional Relation. — Often the relation between the function and the argument can be expressed by a simple formula. For example, if s is the distance fallen from rest in time t, then $s = f(t) = \frac{1}{2}gt^2$.

In such cases, the value of the function for any value taken for the variable can be found by simply substituting in the formula; thus,

$$s_1 = f(1) = \tfrac{1}{2} g, \quad s_2 = f(2) = \tfrac{1}{2} g \cdot 2^2 = 2g,$$

and so for any value.

A function is *tabulated* when values of the argument, as many and as near together as desired, are set down in one column and the corresponding values of the function are set down opposite in another column. For example, in a table of sines, the angle in degrees and minutes is the argument, and the sine of the angle is the function.

A function is *graphed* or exhibited graphically by laying off the values of the argument as abscissas along a horizontal axis, and at the end of each abscissa erecting an ordinate whose length will represent the corresponding value of the function; a curve drawn through the tops (or bottoms) of the ordinates is called the *curve*, or the *graph* of the function.

If $y = f(x)$, the curve is the *locus* of the equation; but it is the length of the ordinate up (or down) to the curve, rather than the curve itself, that represents the function.

If $\rho = f(\theta)$, the function may be graphed by laying off at a point on a line as axis the various angles, — values of the argument θ, and along the terminal sides of the angles the corresponding values of the function ρ; a curve through the ends of the vectorial radii will be the graph of the function, and will be a polar curve. Here, too, it is the length of the radius to the curve, rather than the curve itself, that represents the function. The area under a curve may be taken to represent a function while the ordinate or radius represents some other function. (See Art. 139.)

5. Function — Increasing or Decreasing. — An *increasing* function is one that increases when its variable increases, hence, it decreases when its variable decreases. A *decreasing* function is one that *de*creases when its variable *in*creases, hence it *in*creases when its variable *de*creases. Thus ax and

a^x are increasing, and a/x and $a - x$ are decreasing functions of x.

6. Classes of Functions. — An *algebraic* function is one that without the use of infinite series can be expressed by the operations of addition, subtraction, multiplication, division and the operations denoted by constant exponents.

The common forms are: $(a \pm bx)$, $(a \pm bx^n)$, ax, a/x, x^n, including x^2, x^3, $\sqrt{x}$, $1/\sqrt{x}$, etc.

All functions which are not algebraic are called *transcendental.* Of these, the most important are:

The *exponential functions,* $y = a^x$ or b^x, and $y = e^x$, and their inverse forms, the *logarithmic* functions,

$$x = \log_a y \quad \text{or} \quad \log_b y \quad \text{and} \quad x = \log_e y.$$

The *trigonometric functions,* $y = \sin\theta$, $x = \cos\theta$, $y = \tan\theta$, and the *inverse trigonometric functions,* $\theta = \text{arc}\sin y$ or $\sin^{-1} y$, $\theta = \text{arc}\cos x$ or $\cos^{-1} x$, $\theta = \text{arc}\tan y$ or $\tan^{-1} y$.

The *hyperbolic functions,* $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$; and the *inverse hyperbolic functions,*

$$\sinh^{-1} x = y = \log\left(x + \sqrt{x^2 + 1}\right),$$
$$\cosh^{-1} x = y = \log\left(x \pm \sqrt{x^2 - 1}\right),$$
$$\tanh^{-1} x = y = \tfrac{1}{2}\log\left[(1 + x)/(1 - x)\right].$$

Note. — The phenomena of change in Nature have been held to be generally in accordance with one or the other of three fundamental laws. These have been stated * to be the *parabolic law,* expressed by the power function $y = ax^n$, where n is constant, positive or negative; the *harmonic law,* expressed by the periodic function $y = a\sin(mx)$; and the *law of organic growth,* or the *compound interest law,* expressed by the exponential function $y = ae^{bx}$. It is to be noted that as x increases in arithmetic progression, y of the *exponential* function increases in geometrical progression; while, as x

* In *Elementary Mathematical Analysis,* by Charles S. Slichter.

increases in geometrical progression, y of the *power* function increases in geometrical progression also.

Examples in Nature of the working of these three laws will be given later.

EMPIRICAL EQUATIONS.

Very often the form of a function is given only empirically; that is, the values of the function for certain values of the variable are known from experiment or observation, and the intermediate values are not given; for example, the height of the tide read from a gauge every hour.

In such cases the Calculus is not of much use unless some known mathematical law can be found which represents the function sufficiently accurately.

This problem of finding a mathematical function whose graph shall pass through a series of empirically given points is of great practical importance.

The known values of the function and of the variable are plotted on cross-section paper, "logarithmic squared paper" greatly facilitating the solution, and a smooth curve being drawn "to fit" the determined points, the equation of this curve is required. The curve suggested by the plotted points may have for its equation one of the following forms:

(straight line)	$y = a + bx$, or $y = mx$;
(parabola)	$y = a + bx + cx^2$, or $y = a + cx^2$;
(hyperbola)	$y = a + c/(x + b)$, $y = 1/x$, or $xy = bx + ay$;
(sine curve)	$y = a \sin (bx + c)$, or $y = a \sin (mx)$;
(power function)	$y = ax^n$ (n any number);
(exponential function)	$y = ae^{bx}$.

If the curve suggested by the plotted points is a straight line, determine the values of a and b, or of m, from the observed data. The straight line is not likely to pass through all the points plotted, even when the straight-line

law is the correct expression of the relation to be determined; for the experimental data are subject to error. If the line fits the points within the limits of accuracy of the experiment, it may be drawn through two of the plotted points, and a and b, or m, may be evaluated from their coördinates.

By appropriate treatment of the data many of the laws can be transformed into a linear relation. Thus, when the points plotted suggest a vertical parabola with its vertex on the y-axis, the required equation will be of the form,

$$y = a + cx^2. \tag{1}$$

If t is put for x^2 in (1), and the values of t and y plotted, these values satisfy the relation $y = a + ct$, that is, a straight-line law. The power function $y = ax^n$ may be expressed:

$$\log y = \log a + n \log x, \tag{2}$$

that is, *the logarithms* of the given data satisfy a straight-line law. The straight-line law to fit the logarithms can be determined and compared with (2) to find a and n, which are substituted in $y = ax^n$.

The hyperbolic law and the exponential function also can be transformed to the straight-line law, and the constants evaluated. Whether the experimental data can be expressed by a power function or by an exponential function can be determined by a test. When the data show that, as the argument changes by a constant *factor*, the function also changes by a constant *factor*, then, the relation can be expressed by a power function.

When, however, it is found that a change of the argument by a constant *increment* changes the function by a constant *factor*, then the relation can be expressed by an equation of the exponential type. (See Note, Art. 6.)

A full discussion of this problem of finding the expression of the relation between a function and its argument from limited experimental data involves the theory of least squares, and is out of place in a first course in the Calculus.

This necessarily inadequate treatment of the subject here is warranted by the importance of the problem.

* *Example.* — The amount of water Q, in cu. ft. that flows through 100 feet of pipe of diameter d, in inches, with initial pressure of 50 lbs. per sq. in. is given by the following:

d	1	1.5	2	3	4	6
Q	4.88	13.43	27.50	75.13	152.51	409.54

Find a relation between Q and d.

Let $x = \log d$, $y = \log Q$; then the values of x and y are:

$x = \log d$	0.000	0.176	0.301	0.477	0.602	0.788
$y = \log Q$	0.688	1.128	1.439	1.876	2.183	2.612

These values plotted give points in the xy plane very nearly on a straight line; therefore, taking $y = a + bx$, a and b can be evaluated by measurement on the figure;

$$a = 0.688 = \log 4.88, \quad b = 2.473.$$

Hence, $\log Q = \log 4.88 + 2.473 \log d = \log (4.88\, d^{2.473})$;

whence $Q = 4.88\, d^{2.473}$. * (Ziwet and Hopkins.)

7. Increments. — The amount of change in the value of a variable is called an *increment*. If the variable is increasing, its increment is positive; if it is decreasing, its increment is negative and is really a *de*crement.

An increment of a variable is denoted by putting the letter Δ before it; thus Δx, Δy and $\Delta(x^2)$ denote the increments of x, y, and x^2, respectively. If $y = f(x)$, Δx and Δy denote corresponding increments of x and y, and

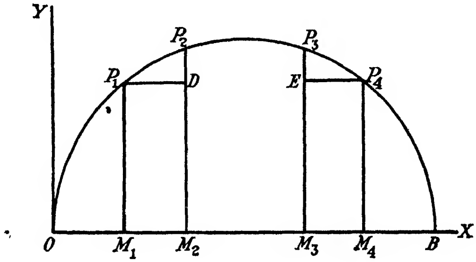
$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x),$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

x denoting any value of x.

In the figure, let $OP_1 \ldots B$ be the locus of $y = f(x)$ referred to the rectangular axes OX and OY. If when $x =$

OM_1, $\Delta x = M_1M_2$, then $\Delta y = M_2P_2 - M_1P_1 = DP_2$; if when $x = OM_3$, $\Delta x = M_3M_4$, then $\Delta y = M_4P_4 - M_3P_3 = -EP_3$.



In the last case Δy is negative and is what algebraically *added* to M_3P_3 gives M_4P_4. When

$$x = OM_1 = x_1, \qquad f(x) = M_1P_1 = f(x_1);$$

when

$$x = OM_2 = x_1 + \Delta x, \quad f(x) = M_2P_2 = f(x_1 + \Delta x);$$

hence when

$$x = x_1, \quad \Delta f(x) = M_2P_2 - M_1P_1 = f(x_1 + \Delta x) - f(x_1).$$

EXERCISE I.

1. One side of a rectangle is 10 feet. Express the variable area A as a function of the other side x.

2. Express the circumference of a circle as a function of its radius r; of its diameter d.

3. Express the area of a circle as a function of its radius r; of its diameter d.

4. Express the diagonal d of a square as a function of a side x.

5. The base of a triangle is 10 feet. Express the variable area A as a function of the altitude y.

6. If
$$y = f(x), \quad y + \Delta y = f(x + \Delta x);$$
$$\therefore \quad \Delta y = \Delta f(x) = f(x + \Delta x) - f(x),$$
and hence,
$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

If $y = mx + b$, find value of Δy and of $\dfrac{\Delta y}{\Delta x}$.

7. If $y = x^2$, find value of Δy and of $\dfrac{\Delta y}{\Delta x}$.

8. If $y = x^3$, find value of Δy and of $\dfrac{\Delta y}{\Delta x}$.

9. If $y = x^n$, find value of Δy and of $\dfrac{\Delta y}{\Delta x}$, assuming the binomial theorem.

10. If $y = f(x) = mx + b$, write values of

$$f(0),\quad f(1),\quad f(-1),\quad f\left(\frac{-b}{m}\right).$$

11. If $f(x) = x^3 - 6x^2 + 11x - 1$, find $f(1) = f(2) = f(3)$.

12. If $\phi(x) = x^3 - 9x^2 + 26x - 14$, find $\phi(2) = \phi(3) = \phi(4) = \frac{1}{4}\phi(5)$.

13. If $f(x) = x^5 - 4x^3 + x$, find $f(-x) = -f(x)$.

14. If $f(x) = x^4 - 6x^2 + 1$, find $f(-x) = f(x)$.

15. If $f(\theta) = \sin\theta + \cos\theta$, find $f(\pi/3)f(2\pi/3) = \frac{1}{2}$.

16. If $f(\theta) = \tan\theta$, find value of $\dfrac{f(\theta) - f(\phi)}{1 + f(\theta)f(\phi)}$.

17. If $f(x) = \log_a x$, find $f(x) - f(y) = f(x/y)$ and $f(x) + f(y) = f(xy)$.

18. If $f(x) = e^x$, find $f(x)\cdot f(y) = f(x + y)$.

19. If $f(x) = \frac{1}{2}(e^x + e^{-x})$, find $f(0)$ and $f(1)$.

20. If $y = f(x) = \dfrac{1 + x}{1 - x}$ and $z = f(y) = f[f(x)]$, find $z = -1/x$.

8. Uniform and Non-uniform Change. — When the ratio of the corresponding increments of two variables is *constant*, either variable is said to change uniformly with respect to the other.

When $y = mx + b$, $\dfrac{\Delta y}{\Delta x} = m$ (constant). (6, *Exercise* I.)

It follows that any linear function of x changes uniformly with respect to x; that is, y changes uniformly with respect to x when the point (x, y) moves along any straight line.

When the ratio of the corresponding increments of two variables is *variable*, either variable is said to change non-uniformly with respect to the other.

When $y = x^2$, $\dfrac{\Delta y}{\Delta x} = 2x + \Delta x$ (variable). (7, *Exercise* I.)
Thus the area of a square changes non-uniformly with respect to a side. Any non-linear function of x changes non-uniformly with respect to x, for evidently y changes non-uniformly with respect to x when the point (x, y) moves along any curved line.

Since time changes uniformly, any variable will change uniformly when it receives *equal* increments in *equal* times; and it will change non-uniformly when it receives *unequal* increments in *equal* times.

Thus in $s = vt$, where s is the space passed over in time t by an object moving with constant velocity v, s changes uniformly.

In $s = \frac{1}{2} gt^2$, where the object moves with constant acceleration g, s changes non-uniformly.

9. Differentials. — The *differentials* of variables that change uniformly with respect to each other are their corresponding increments; that is, their actual changes.

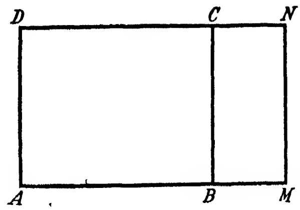
The *differentials* of variables that change non-uniformly with respect to each other are what *would be* their corresponding increments if, at the corresponding values considered, the change of each became and continued uniform.

As with increments, the differentials will be positive or negative according as the variables are increasing or decreasing.

The differential of a variable is denoted by putting the letter d before it; thus, dx, read "differential x," is the symbol for the differential of x. The differential of a variable or function consisting of more than a single letter is indicated by the letter d before a parenthesis enclosing the variable or function; thus, $d(x^2)$, $d(mx + b)$, $d(f(x))$, denote the differentials of x^2, $mx + b$, and $f(x)$, respectively.

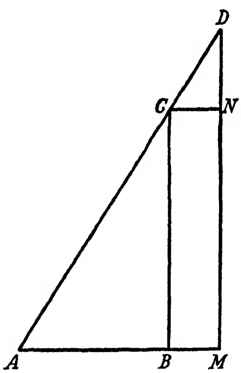
10. Illustrations of Differentials. — (a) Suppose a rectangle, with constant altitude, is changing by the base in-

creasing. If when the base is AB its increase is BM, then d (base) $= BM$, and d (rectangle) $= BMNC$.



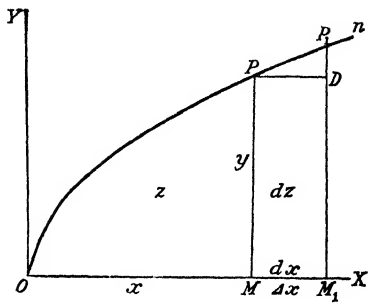
Here the variables change uniformly with respect to each other, hence their differentials are their corresponding increments.

(*b*) Conceive a right triangle, with variable base and altitude, is changing by the altitude moving uniformly to the right. If when the base is AB its increment is BM, then the increment of the triangle will be $BMDC$. But if the increase of the triangle became uniform at the value ABC, the increment of the triangle in the same time would evidently be $BMNC$; hence, $BMNC$ and BM may be taken as the differentials of the triangle and of the base, where the base is AB.



In this case the triangle changes non-uniformly with respect to its base, so its differential is what *would be* its increment if, at the value considered, the change became uniform. Since the base changes uniformly, its differential is its actual increment. Here increment of triangle $ABC =$ d (triangle ABC) $+$ triangle CND, while Δ (base) $= d$ (base). If the change of a variable be *uniform*, any *actual* increment may be taken as its differential. If time be considered, the interval of time, though arbitrary, must be the same for a function as for its variable.

(*c*) Let the curve OPn be the locus of $y = f(x)$, referred to the axes OX and OY. Conceive the area between OX



and the curve as traced by the ordinate of the curve moving uniformly to the right. Let z denote this area, and let MM_1 be Δx reckoned from the value $OM = x$; then $MM_1P_1P = \Delta z$. But if the increase of z became uniform at the value OMP, its increment in the same interval would be MM_1DP; hence MM_1 and MM_1DP may be taken as the differentials of x and z respectively, when $x = OM$.

Hence $$dz = MM_1DP = MP\,dx = y\,dx,$$

which shows that area z is changing y times as much as x

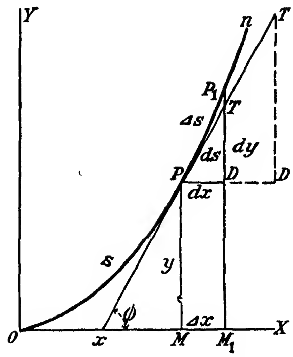
Here $$\Delta z = dz + \text{area } PDP_1.$$

It is seen here that while the actual change in the area does not admit of an exact geometrical expression, the *differential* of the area, being a rectangle, is exactly and simply expressed. It will be shown further on how by *Integration* an exact expression for the area itself is obtained from this expression for the differential.

Note. — Historically the Calculus originated through the efforts to obtain the exact area of figures bounded by curves, mathematics up to that time having furnished no method applicable to all curves whose equations were known.

It is true too that historically the method of Integration was discovered before the method of Differentiation was developed. The Differential Calculus arose through the problem of determining the direction of the tangent at any point of a curve. (See Note, Art. 75.)

(*d*) Let OPn be the locus of $y = f(x)$ and s the length from O along the curve. Suppose the point (x, y) to move



along the curve to P and thence along the tangent at that point. Then at the value $x = OM$, the change of x and y would become uniform with respect to each other, as the point (x, y) would be moving along a straight line. The change of s would become uniform also with respect to both x and y. As x is the independent variable it may be taken to vary uniformly, making PD or $dx = \Delta x$ or MM_1, the *actual* change in x as the point moves along the *curve* from P to P_1. Then dy is DT, the corresponding *uniform* change of y, and ds is PT, the corresponding *uniform* change of s. It is evident that while $dx = \Delta x$, dy is *not* equal to Δy and ds is *not* equal to Δs. When, and *only* when, the locus is a *straight* line will $dy = \Delta y$ and $ds = \Delta s$, after dx has been taken equal to Δx.

It should be noted that it is not essential that dx should be made equal to Δx, for dx may be taken as any value other than zero, and then dy will be the perpendicular distance from the end of dx to the tangent and ds will be the distance from the point (x, y) along the tangent to end of dy. From figure, $(ds)^2 = (dx)^2 + (dy)^2$.

11. Rate, Slope, and Velocity. — The differential triangle PDT in figure for (*d*) Art. 10, gives $\dfrac{dy}{dx} = \tan\phi =$ slope of the curve $y = f(x)$ at point (x, y), and $\dfrac{dy}{dx}$ is the ratio of the change of y to the change of x at the point (x, y), or for any corresponding values of x and y; and $\dfrac{dy}{dx}$ is called the *rate* of y with respect to x.

$\dfrac{\Delta y}{\Delta x}$ is the *average* slope or the *average* rate of change of y

with respect to x, while the point (x, y) moves over Δs on the curve or while x and y take successive values over any range.

If $s = f(t)$, where s denotes distance from some origin, and t, the time elapsed, then $\frac{ds}{dt}$ is the rate of change of s with respect to t, what is called velocity, speed, or rate of motion: $v = \frac{ds}{dt}$.

In the case of uniform motion in a straight or curved path, $v = \frac{s}{t} = \frac{\Delta s}{\Delta t} = \frac{ds}{dt} =$ a constant. In the case of non-uniform or variable motion, $v = \frac{ds}{dt} =$ a variable.

In figure for (*d*) Art. 10, it is seen that $(ds)^2 = (dx)^2 + (dy)^2$; dividing by $(dt)^2$, $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$;

$\therefore$ velocity of a point in its path is resultant velocity,

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{v_x^2 + v_y^2},$$

x-component is $v_x = \frac{dx}{dt} =$ velocity parallel to x-axis,

y-component is $v_y = \frac{dy}{dt} =$ velocity parallel to y-axis;

$$\tan\phi = \frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}; \quad \therefore \quad \frac{dy}{dt} = \frac{dx}{dt}\tan\phi, \text{ or } v_y = v_x \tan\phi;$$

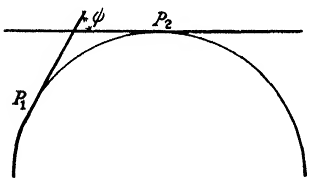
$$\sin\phi = \frac{dy}{ds} = \frac{dy}{dt} \bigg/ \frac{ds}{dt}; \quad \therefore \quad \frac{dy}{dt} = \frac{ds}{dt}\sin\phi, \text{ or } v_y = v\sin\phi;$$

$$\cos\phi = \frac{dx}{ds} = \frac{dx}{dt} \bigg/ \frac{ds}{dt}; \quad \therefore \quad \frac{dx}{dt} = \frac{ds}{dt}\cos\phi, \text{ or } v_x = v\cos\phi.$$

It appears that $\frac{dy}{dx}$, the rate of y with respect to x, is the ratio of the time rate of y to the time rate of x.

These expressions for velocity and their relations include the case in which the motion is uniform or variable along a straight line.

12. Rate, Speed, and Acceleration. — *Acceleration* is rate of change of speed or velocity. Hence, if the speed is changing, $\frac{dv}{dt}$, the time rate of change of speed, is called the acceleration along the path, or the tangential acceleration, and will be denoted by a_t. The total acceleration a is equal to a_t, when the path is a straight line; otherwise, they are not equal. It is desirable to distinguish between speed and velocity. A body is in motion relative to some other body when its position is changing with respect to that other. Change of position involves change of distance or of direction or of both distance and direction. If a point moves continuously in the same direction, the path is a straight line; if the direction is continuously changing, the path is a curved line. The direction of motion at any point of a curvilinear path is the direction of the tangent at that point, and from one point to another the direction of motion changes through the angle between the two tangents. Thus from P_1 to P_2 the direction changes through angle ϕ. When the position of a point changes the displacement takes place along some continuous path, straight or curved, and a certain time elapses. The rate at which the change of position takes place is the *velocity* of the point.



If the point moves so that equal distances are passed over in equal intervals of time, the motion is uniform and the point has *constant speed*, whether the path is straight or curved. If the direction also is constant, that is, if the path is a straight line, the point has *constant velocity*. Thus there is uniform motion with constant speed either in a

straight line or in a curved line, but there is uniform or constant velocity in a straight line *only*.

The extremity of either hand of a clock moves in a circular path over equal distances in equal intervals of time, but its direction is continuously changing. The motion is uniform and the *speed* constant, but the *velocity* is not constant since the direction is variable. Hence, a body may move in a circle with constant speed and yet its velocity is variable. In this case the acceleration a_t along the tangent is zero, while the total acceleration a, the rate of change of the *velocity*, is normal, directed towards the center, and has a constant value depending upon the speed and radius. (This value will be derived later.)

The term speed thus denotes the magnitude of a velocity. However, the term velocity itself is ordinarily used in the sense of speed as well as in the strict sense of speed and direction. In the great majority of cases the direction is assumed to be known, and the magnitude of the velocity is what is in question.

Note. — A velocity having both magnitude and direction is what is called a *vector* quantity and can be represented by a straight line having the direction of the velocity and a length denoting its magnitude. Hence the sides of the triangle PTD in figure for (*d*) Art. 10, may be taken to represent the resultant velocity v and its components v_x and v_y.

13. Rate and Flexion. — *Flexion* has been adopted by some writers as a term for the rate of change of slope. Hence, when the slope changes, and it always does except for a straight line, $\frac{dm}{dx}$, the rate of change of the slope with respect to x will be called the flexion of the curve and will be denoted by b, from the word *bend*. When the velocity and the slope are uniform, there is no acceleration and no flexion; that is, $\frac{dv}{dt} = 0$ and $\frac{dm}{dx} = 0$.

14. Illustrations. — Consider the established equations of motion:

$s = vt$ or $v = \frac{s}{t}$;

$v = gt = 32t$ ($a_t = g = 32$ ft. per sec. per sec. approximately);

$s = \frac{1}{2}gt^2 = 16\,t^2$.

When the motion is uniform the velocity or speed is the whole distance divided by the whole time; or any increment of the distance divided by the corresponding increment of the time is the velocity at any point, and it is the same as at any other point, since it is constant.

$$\therefore \quad v = \frac{s}{t} = \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \text{a constant.}$$

In the case of variable motion the whole distance divided by the whole time gives the *average* velocity over the whole distance; or any increment of the distance divided by the corresponding increment of the time gives the *average* velocity over that increment of the distance. The velocity at any point is now given by the distance that *would* be gone over in any time divided by that time, if at the point the motion became and continued uniform or the velocity became constant.

Thus $v = \frac{ds}{dt} = 32\,t$ gives $v = 32$ ft. per sec. at the end of the first second; and means that the distance in the next second *would* be 32 ft., if at the end of the first second the velocity became constant.

As a matter of fact, $s = 16\,t^2$ gives 16 feet for the distance in the first second, and 48 feet for the distance actually passed over in the next second. This variation of distance is of course due to the velocity being constantly accelerated.

So when it is said that a train at any point is moving at sixty miles per hour, it is not asserted that it will actually go sixty miles in the next hour; but what is implied is, that the train *would* go sixty miles in any hour if from that point it continued to move with unchanged velocity.

Therefore, in ordinary language, variable velocity is expressed by the differential of the distance divided by the differential of the time; that is, by $\frac{ds}{dt}$.

In the case above,

$$\frac{ds}{dt} = \frac{60 \text{ miles}}{1 \text{ hour}} = \frac{1 \text{ mile}}{1 \text{ min.}} = \frac{88 \text{ feet}}{1 \text{ sec.}};$$

thus dt may be taken as any value other than zero, if the corresponding value of ds is taken.

EXERCISE II.

1. $u = 2x$. Show graphically the change in u when x is given an increment, by taking x as the base of a variable rectangle of altitude 2, and u as the area. Is the change uniform for u?

2. $u = x^2$. (*a*) Show graphically the change in u when x is given an increment, by taking x as the side of a variable square, and u as the area. Show graphically the change in u if the change became uniform. (*b*) Show same when x is taken as the base of a variable right triangle of altitude $2x$, and u as the area. Show the change in u if the change became uniform.

3. $V = x^3$. Show graphically the change in V when x is given an increment, if the change in V became uniform; x being the side of a variable cube, and V the volume of the cube.

4. If a body is moving with uniform velocity and passes over 1000 feet in 10 seconds, what is its velocity at any point? If distance is taken as axis of ordinates and time as axis of abscissas, what would the slope of the graph be?

5. $s = 16t^2$. Compute the values of s when $t = 1, 2, 1.1, 1.01, 1.001$.

Get the average velocity between $t = 1$ and $t = 2$, between $t = 1$ and $t = 1.1$, between $t = 1$ and $t = 1.01$, between $t = 1$ and $t = 1.001$.

From $v = 32t$, get the velocity at $t = 1$ and compare average velocities with it. What would be the distance passed over in the

second second, if at the end of the first the velocity became uniform? What is the actual distance passed over in the second second? Which is the increment? Which is the differential?

6. If a ship is sailing northeast at 10 knots, what is its northerly rate of motion? What is its easterly rate?

If it is sailing S. 30° W. at 10 knots, what is its southerly rate? What is its westerly rate?

7. If the grade of a road is such that the rise is 52.8 feet in every mile, what is the slope?

8. If the grade of a road is continuously changing, the average slope is given by what? The slope at any point would be the slope of what?

CHAPTER II.

DIFFERENTIATION. DERIVATIVES. LIMITS.

15. Derivative. — The ratio of the differential of a function of a single variable to that of the variable is called the *derivative* of the function. Thus $\frac{dy}{dx}$ denotes the derivative of y as a function of x. Since the derivative may vary with x, as the slope of a curve varies from point to point, it is, in general, itself a function of x; hence, the derivative of $f(x)$ is appropriately denoted by $f'(x)$, and is often called the *derived* function. So

$$\frac{df(x)}{dx} \equiv f'(x).$$

If
$$y = f(x),$$
$$\frac{dy}{dx} = f'(x);$$
$$\therefore \quad dy = f'(x)\,dx.$$

Since $dy = f'(x)\,dx$, the derivative is also called the differential coefficient. The derivative $\frac{dy}{dx}$ is sometimes denoted by $D_x y$.

In the case of a curve the *derivative* is the *slope*, in the case of motion it is the *velocity, speed,* or *rate* of *motion*; in every case it is the *rate of change* of the function with respect to the argument or variable.

Examples. —

If $y = f(x) = mx + b$, the derivative $\frac{dy}{dx} = f'(x) = m$.

If $s = f(t) = vt$, the derivative $\dfrac{ds}{dt} = f'(t) = v$.

If $v = f(t) = gt$, the derivative $\dfrac{dv}{dt} = f'(t) = g$.

Here m, v, and g are constants.

16. Differentiation. — The operation of finding the differential of the function in terms of the differential of the argument, or the equivalent operation of finding the derivative, is called *differentiation*. The sign of differentiation is the letter d; thus d in the expression $d(x^2)$ indicates the operation of finding the differential of x^2, and in $\dfrac{d}{dx}(x^2)$, that of finding the derivative. D_xy, $\dfrac{dy}{dx}$, and $f'(x)$ each denote the derivative of y as a function of x.

The general method of getting the derivative of $y = f(x)$ is by finding the limit of the ratio of the increments of y and x as they are diminished towards zero as a limit; for the limit which the ratio approaches, when defined to be the *derivative*, can be shown to be $\dfrac{dy}{dx}$.

17. Limits. — The student has been made acquainted in Geometry with the notion and use of limits; for example, the area or the circumference of the circle, as the limit of the area or the perimeter of the inscribed and circumscribed polygons, when the number of sides increases without limit, or when the length of the side approaches zero as a limit. A precise statement of a limit as used in the Calculus is as follows:

When the difference between a variable and a constant becomes and remains less, in absolute value, than any assigned positive quantity, however small, then the constant is the limit of the variable.

If x is the variable and a is the limit, the notation is $\lim x = a$, or $x \doteq a$, or $\lim (a - x) = 0$, or $(a - x) \doteq 0$;

in which $\doteq$ is the symbol for *approaches as a limit.* When the limit of a variable is zero, the variable is an *infinitesimal.* The difference between any variable and its limit is always an infinitesimal.

18. Theorems of Limits. — The elementary theorems of limits are:

1. If two variables are equal, their limits are equal.

2. The limit of the sum or product of a constant and a variable is the sum or product of the constant and the limit of the variable.

3. The limit of the variable sum or product of two or more variables is the sum or product of their limits.

4. The limit of the variable quotient of two variables is the quotient of their limits, *except when the limit of the divisor is zero.* (See Note, Art. 20, for proof.)

Note. — The Differential Calculus solves such limits as the exceptional case just stated.

19. Derivative as a Limit. — The limit of a variable, as z, is often written $lt\,(z)$.

$$\operatorname*{Lim}_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right], \text{ or } lt\frac{\Delta y}{\Delta x}, \text{ denotes } lt\left(\frac{\Delta y}{\Delta x}\right) \text{ when } \Delta x \doteq 0.$$

In defining $\frac{dy}{dx}$ as a rate (Art. 11), it is stated that $\frac{\Delta y}{\Delta x}$ is the average slope, or average rate of change of y with respect to x, over the range Δx. As has been given (Art. 7),

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

where $y = f(x)$ and x is any value of x. It remains to be shown that

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = f'(x).$$

(a) *By rates without the aid of a locus.* Let time rates be used and let t, x, and y denote *any* corresponding values of

t, x, and y, from which Δt, Δx, and Δy are reckoned. Since $\frac{\Delta y}{\Delta t}$ is the average rate of y over interval Δy, $\frac{\Delta y}{\Delta t}$ is the time rate of y at a value of y between y and $y + \Delta y$;

$$\therefore \quad lt\frac{\Delta y}{\Delta t} = \left\{\begin{array}{c}\text{time rate of } y \text{ at the value of } y\\ \text{from which } \Delta y \text{ is reckoned.}\end{array}\right\} \tag{1}$$

Also

$$lt\frac{\Delta x}{\Delta t} = \left\{\begin{array}{c}\text{time rate of } x \text{ at the value of } x\\ \text{from which } \Delta x \text{ is reckoned.}\end{array}\right\} \tag{2}$$

Dividing (1) by (2), there results,

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \frac{\text{the time-rate of } y}{\text{the time-rate of } x} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{dy}{dx}.$$

(Compare Art. 11.)

Thus in showing the derivative as a limit, it appears that the derivative of a function expresses the ratio of the rate of change of the function to that of its variable. It is evident that a function is an increasing or a decreasing function according as its derivative is positive or negative.

In the above derivation in place of time-rates, the rate of any other variable of which x and y are functions could be used.

Remarks. Function of a Function. — It should be noted that x and y being taken as functions of a third variable t, to every value of this auxiliary variable there corresponds a *pair* of values of x and y, so y is indirectly determined as a function of x. The derivative of y as a function of x *mediately* through t is, as shown;

$$\frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt}.$$

Solving gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt},$$

and this gives the formula for the derivative of the function of a *function*. For if y is directly given as a function of x, and x as a function of t, then y is said to

be a function of a function of t, as it is given as a function of t *mediately* through x. If y is a given function of z, and z a given function of x, then y is a function of a function of x, and the formula for the derivative of y is

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Functions of functions often occur and there may be several intermediate variables such as z in above case. A function, as $f(x)$, is defined to be continuous for the value a of x, or, more simply, continuous *at* a, if $f(a)$ is a definite finite number, and if $\lim\limits_{x \doteq a} f(x) = f(a)$; that is, if $\lim f(x) = f(\lim x)$. By this definition the elementary functions of a single variable are continuous for all values of the variable except those for which a function becomes infinite.

A concrete case in everyday experience of a function of a function is the change in length of a metal bar as the temperature changes with time. Here the length is a function of the temperature, and the temperature is a function of the time; hence the length is a function of a function of the time. The length, being directly a function of the temperature, is indirectly a function of the time through the temperature, which is directly a function of the time.

The rate of change of length per second is equal to the product of the rate of change of length per degree and the rate of change of temperature per second. If l, T, and t denote the length, temperature, and time, respectively, then, in accordance with the formula,

$$\frac{dl}{dt} = \frac{dl}{dT} \cdot \frac{dT}{dt}.$$

If the length and temperature are taken as changing each directly with the time, then the rate of change of the length per degree is equal to the rate of change of length per second divided by the rate of change of temperature per second.

The formula would be

$$\frac{dl}{dT} = \frac{dl}{dt} \Big/ \frac{dT}{dt},$$

which may be obtained from the other formula by solving; or the first may be obtained from this. As all variables change with time, that is, are functions of time, time rates are most common.

(*b*) To show geometrically

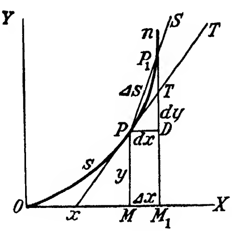
$$\lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx} = \textbf{slope of curve.}$$

Let OPn be the locus of $y = f(x)$, PP_1S a secant, and PT a tangent at P. If arc $OP = s$, arc $PP_1 = \Delta s$. Let

$$OM = x, \quad MM_1 = \Delta x, \quad \text{then} \quad MP = y, \quad DP_1 = \Delta y.$$

Hence

$$\frac{\Delta y}{\Delta x} = \frac{DP_1}{PD} = \text{slope of secant } PP_1S.$$



Conceive the secant PP_1S to be revolved about P so that arc PP_1 $(= \Delta s) \doteq 0$; then $\Delta x \doteq 0$, $\Delta y \doteq 0$, and the slope of the secant $\doteq$ the slope of the tangent at P.

Hence $\lim\limits_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx} =$ slope of the curve $y = f(x)$ at point (x, y).

The *limit* is thus shown to be the *derivative*, whether $\frac{dy}{dx}$ is regarded as only a *symbol* for the limit of the ratio of the increments of y and x, or as that *limit* and also a definite *ratio* itself of the *differentials* of y and x.

Corollary. — If when $\Delta x \doteq 0$, $\frac{\Delta y}{\Delta x}$ varies, the locus of $y = f(x)$ is a curved line; otherwise, if $\frac{\Delta y}{\Delta x}$ is constant, the

locus is a straight line coincident with the tangent, and $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$. So for a straight line, the ratio of the increments of y and x, being constant, does not approach a limit as Δx approaches zero, and the derivative is the constant ratio $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$.

In general, $\frac{\Delta y}{\Delta x}$ will approach a finite limit except where the locus is perpendicular or parallel to the x-axis, when the slope is infinite or zero. On special curves where there are two tangents at a point, the limit is not definite. (See Note, Art. 80.)

Note. — This definition of the derivative as the limit of the ratio of the increments of y and x as they converge towards zero is the fundamental conception of the Differential Calculus by the method of limits.

In this method since Δy and Δx are variables each approaching zero as a limit, they are infinitesimals, for any variable with zero as a limit is defined to be an infinitesimal. If dx is taken as *always* equal to Δx, then, except when the locus is a straight line, dy will *always* differ from Δy; and dx, dy, Δy and $(\Delta y - dy)$ are infinitesimals when Δx approaches zero, for they each approach zero as Δx approaches zero. However, dx may be taken as *any* increment of x and, when x is the independent variable, may be made a finite *constant*, for x may be considered as changing uniformly by finite increments; but then, except for a *straight* line, dy is *variable* though finite. When Δx is infinitesimal any particular value of Δx may be taken as constant, for any particular value of an infinitesimal is a fixed finite quantity, small or large as the case may be. Whether dy and dx are infinitesimals or finite quantities, their ratio for any particular value of the variable is, in general, *constant;* and it is their *ratio* that is important.

Both ways of regarding differentials are useful. Finite

differentials are desirable for their simplicity, especially to make the differential of the independent variable constant. But when *Integration* is regarded as finding the *limit of a sum*, as will be shown later, differentials are necessarily infinitesimal.

One advantage in making dx infinitesimal and taking it very small is that Δy is then very nearly equal to dy, and so instead of computing Δy in some investigation the simpler and easily found dy may be taken for it. In practical work dx and dy are usually taken very small quantities, but it is their *ratio* that is of importance. In this connection it should be borne in mind that, however small a quantity may be, it is not an infinitesimal as defined in the Calculus, unless it is a *variable* approaching zero as a limit.

20. Illustrative Examples. — In these examples, as elsewhere, the letter symbol for the argument in general is used as the symbol also for some particular value of the argument; this double use of the symbol making for conciseness and generality.

Example 1. — Let the function to be differentiated be

$$y = x^2. \tag{1}$$

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\,\Delta x + \overline{\Delta x}^2, \tag{2}$$

$$\Delta y = 2x\Delta x + \overline{\Delta x}^2, \tag{3}$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x, \tag{4}$$

$$m = \frac{dy}{dx} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}(2x + \Delta x) = 2x; \tag{5}$$

$$\therefore \quad dy = 2x\,dx. \tag{6}$$

The actual change of y corresponding to any change of x is given by (3). The average rate of change of y from any value of x to $x + \Delta x$, or the average slope of the curve over that range, is given by (4). The rate of change of y with respect to x at any value of x, or the slope of the curve at

any point (x, y), is given by (5). What *would* be the change of y for any change of x, if at any value of x the change of y became uniform, is given by (6); and it shows that, at any point (x, y), y is changing $2x$ times as much as x is changing.

Example 2. — Let the function to be differentiated be

$$s = 16t^2. \tag{1}$$

$$s + \Delta s = 16(t + \Delta t)^2 = 16\left(t^2 + 2t\,\Delta t + \overline{\Delta t}^2\right), \tag{2}$$

$$\Delta s = 32t\,\Delta t + 16\,\overline{\Delta t}^2, \tag{3}$$

$$\frac{\Delta s}{\Delta t} = 32t + 16\,\Delta t, \tag{4}$$

$$v = \frac{ds}{dt} = \lim_{\Delta t \doteq 0}\left[\frac{\Delta s}{\Delta t}\right] = 32t; \tag{5}$$

$$\therefore \quad ds = 32t\,dt. \tag{6}$$

Let the function to be differentiated be

$$v = 32t. \tag{5}$$

$$v + \Delta v = 32(t + \Delta t), \tag{7}$$

$$\Delta v = 32\,\Delta t, \tag{8}$$

$$\frac{\Delta v}{\Delta t} = 32, \tag{9}$$

$$a = a_t = \frac{dv}{dt} = \frac{\Delta v}{\Delta t} = 32. \tag{10}$$

The distance s passed over by a body falling from rest in any time t is given by (1). The actual distance passed over in time Δt, after any time t, is given by (3). The *average* time-rate of the distance, or the *average* velocity from s to $s + \Delta s$, is given by (4). The time-rate of the distance, or the velocity at end of any time t, is given by (5). What *would* be the distance passed over in time dt $(= \Delta t)$, if at end of time t the body moved on with unchanged velocity, is given by (6). The actual change of the velocity in time Δt is given by (8). That the velocity is changing uniformly is shown by (9), since the ratio of the two increments is constant.

The rate of change of the velocity or speed, the acceleration a or a_t, is given and shown to be constant by (10).

It is to be noted that $\frac{\Delta v}{\Delta t}$, being a constant ratio by (9), does not approach a limit; hence, the derivative $\frac{dv}{dt}$ is equal to $\frac{\Delta v}{\Delta t}$ and is therefore constant acceleration. (See *Corollary*, (*b*), Art. 19.)

Note. — If $s = 16\,t^2$ be represented graphically by a curve, then the slope of the curve is $m = \frac{ds}{dt} = 32\,t = v$, the velocity; and the flexion of the curve is $b = \frac{dm}{dt} = \frac{dv}{dt} = 32 = a_t$, the acceleration.

Example 3. — Let the function to be differentiated be:

$$y = mx + b. \tag{1}$$

$$y + \Delta y = m\,(x + \Delta x) + b = mx + m \cdot \Delta x + b, \tag{2}$$

$$\Delta y = m \cdot \Delta x, \tag{3}$$

$$\frac{\Delta y}{\Delta x} = m, \tag{4}$$

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = m, \tag{5}$$

$$\therefore \quad dy = m\,dx. \tag{6}$$

Here again the ratio of the increments, being shown by (4) to be a constant m, does not approach a limit; hence, as shown by (5) the derivative $\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = m$, the constant slope of the line $y = mx + b$.

That the ordinate is changing m times as much as the abscissa is shown by (6).

It is evident that for a linear function not only is the *ratio* of the increments the derivative, but the *increments* are the *differentials* as defined.

Example 4. — Let the function to be differentiated be

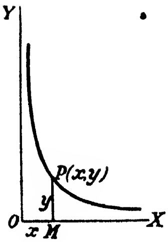
$$y = \frac{1}{x} \quad \text{or} \quad xy = 1. \tag{1}$$

$$(x + \Delta x)(y + \Delta y) = xy + x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y = 1, \tag{2}$$

$$x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y = 0, \quad \text{or} \quad (x + \Delta x)\,\Delta y = -y\,\Delta x, \tag{3}$$

$$\frac{\Delta y}{\Delta x} = -\frac{y}{x + \Delta x}, \text{ average slope over } \Delta x, \tag{4}$$

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left(-\frac{y}{x + \Delta x}\right) = -\frac{y}{x} = \tan\phi,$$

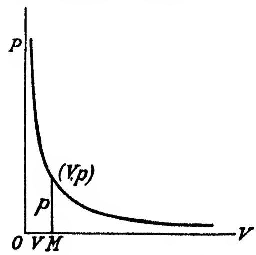


slope at any point (x, y); (5)

$$\therefore \quad dy = -\frac{y}{x}\,dx, \tag{6}$$

showing that the $\dfrac{\textit{decrease}}{\textit{increase}}$ of y is $\dfrac{y}{x}$ times the $\dfrac{\textit{increase}}{\textit{decrease}}$ of x, at any point (x, y).

Example 5. — In compressing air, if the temperature of the air is kept constant, the pressure and the volume are connected by the relation $pV =$ constant. To find the rate of change of the pressure with respect to the volume, that is, the derivative $\dfrac{dp}{dV}$.



Let $pV = K$, (1)

$$(p + \Delta p)(V + \Delta V) = pV + p\,\Delta V + V\,\Delta p + \Delta p\,\Delta V = K, \tag{2}$$

$$p\,\Delta V + V\,\Delta p + \Delta p\,\Delta V = 0, \text{ or } (V + \Delta V)\,\Delta p = -p\,\Delta V; \tag{3}$$

$$\frac{\Delta p}{\Delta V} = -\frac{p}{V + \Delta V}, \tag{4}$$

average rate of change from V to $V + \Delta V$,

$$\frac{dp}{dV} = \lim_{\Delta V \doteq 0}\left[\frac{\Delta p}{\Delta V}\right] = \lim_{\Delta V \doteq 0}\left(-\frac{p}{V + \Delta V}\right) = -\frac{p}{V}, \tag{5}$$

rate of change for any corresponding values of p and V;

$\therefore \quad dp = -\frac{p}{V}dV$, showing that the $\frac{\textit{decrease}}{\textit{increase}}$ of pressure is $\frac{p}{V}$ times the $\frac{\textit{increase}}{\textit{decrease}}$ of volume at any corresponding values of pressure and volume.

Example 6. — Let M be the mass of a body, V its volume, and ρ its density; then,

$$\frac{\Delta M}{\Delta V} = \frac{M}{V} = \rho,$$

the density at any point, when the body is of uniform density;

$$\lim_{\Delta V \doteq 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV} = \rho,$$

the density at any point, when the density varies from point to point. Here when the body is not homogeneous, the density being variable, $\frac{\Delta M}{\Delta V}$ is the average density of the portion of mass, ΔM; while the derivative, $\frac{dM}{dV}$, expresses the density at a point of the body whether the density is *variable* or *uniform*.

Note. — In regard to $\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx}$, the derivative of y as a function of x, it is important to note that, since the limit of the divisor is zero, it is wrong to write

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \frac{\lim \Delta y}{\lim \Delta x} = \frac{0}{0}.$$

This case is specially excepted in Theorem 4, Art. 18. *To prove the Theorem* 4, Art. 18:

Since $\qquad y = \frac{y}{x}\cdot x,$

$$\lim y = \lim \frac{y}{x} \cdot \lim x, \text{ by Theorem 3,}$$

$$\therefore \quad \lim \frac{y}{x} = \frac{\lim y}{\lim x}, \textit{ if lim } x \textit{ is not zero.}$$

When $\lim x$ is zero, division by it is inadmissible by the laws of Algebra. If $\lim x$ were zero and $\lim y$ not zero, then $\frac{y}{x}$ is infinite and has no limit; hence the exception in Theorem 4. The notation $\lim\limits_{x \doteq 0} \frac{y}{x} = \infty$, if so written, means that, as x approaches zero as a limit, $\frac{y}{x}$ increases without limit; that is, the limit is non-existent.

Infinity or an infinite quantity is not a limit, and the symbol ∞ means a *variable* increasing without *limit.*

In Example 4, where $y = \frac{1}{x}$, $\frac{y}{x} = \frac{1}{x^2}$. Here where $\lim x$ is zero and $\lim y$ is *not* zero, $\frac{y}{x}$ is infinite, having no limit.

In Example 5, the limit $-\frac{p}{V}$ is finite for finite values of p and V. From $p = \frac{K}{V}$ and $\frac{p}{V} = \frac{K}{V^2}$, $p = \infty$ as $V \doteq 0$; hence as $\lim V$ is zero and $\lim p$ is *not* zero, $\frac{p}{V}$ is infinite;

$$\therefore \quad \frac{\Delta p}{\Delta V} = -\frac{p}{V + \Delta V} \text{ is infinite as } V \doteq 0,$$

and $\lim\limits_{\Delta V \doteq 0} \left[\frac{\Delta p}{\Delta V}\right]$ is non-existent when V is infinitesimal.

In Example 3, where $y = mx + b$, $\Delta y = m\,\Delta x$, and $\frac{\Delta y}{\Delta x} = m$.

If $\Delta x \doteq 0$, $\Delta y \doteq 0$, but their ratio is constant and approaches no limit. Since $\Delta y = m\Delta x$, the law of change of the variables is known and the ratio of two infinitesimals is a finite constant.

In Example 1, where $y = x^2$,

$$\Delta y = 2\,x\,\Delta x + \overline{\Delta x}^2, \quad \frac{\Delta y}{\Delta x} = 2\,x + \Delta x, \quad \therefore \quad \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = 2\,x.$$

Here the limit of the ratio of two infinitesimals is a finite constant for any particular finite value of x; but, as x may

have *any* value, the limit of the ratio may be zero, finite, or non-existent.

Thus it is seen that, no matter how small two quantities may be, their ratio may be either small or large; and that, if the two quantities are variables both with zero as their limit, the limit of their ratio may be either finite, zero, or non-existent, but is *not* 0/0. (See Art. 160.)

To find $\lim\limits_{\Delta x \doteq 0}\left[\dfrac{\Delta y}{\Delta x}\right]$, as in the illustrative examples, the limit of an equal variable is found; which limit is, in general, *determinate* and not identical with the indeterminate expression 0/0. In certain cases the limit of the ratio of two infinitesimals is found by finding the limit of some other variable which, though not equal to the ratio, has the same limit. Examples of such cases will be given further on. (See Art. 70, Art. 77.)

21. Replacement Theorem. — *The limit of the ratio of two variables is the same when either variable is replaced by any other variable the limit of whose ratio to the one replaced is unity.*

Let θ, θ_1, ϕ, and ϕ_1, be any four variables, so that

$$lt\,\frac{\theta}{\theta_1} = 1, \qquad lt\,\frac{\phi}{\phi_1} = 1, \quad \text{and} \quad lt\,\frac{\theta}{\phi} = c. \tag{1}$$

$$\frac{\theta}{\phi} = \frac{\theta}{\phi}\cdot\frac{\theta_1}{\theta_1}\cdot\frac{\phi_1}{\phi_1} = \frac{\theta_1}{\phi_1}\cdot\frac{\theta}{\theta_1}\cdot\frac{\phi_1}{\phi};$$

$$\therefore \quad lt\,\frac{\theta}{\phi} = lt\,\frac{\theta_1}{\phi_1}\cdot lt\,\frac{\theta}{\theta_1}\cdot lt\,\frac{\phi_1}{\phi} = lt\,\frac{\theta_1}{\phi_1}, \qquad \text{by (1)}$$

in which θ is replaced by θ_1, and ϕ by ϕ_1, but the limit of the two variables is the same.

22. Limit of Infinitesimal Arc and Chord. — *The limit of the ratio of an infinitesimal arc of any plane curve to its chord is unity.*

Since s (Art. 19 (*b*), figure) is a function of x,

$$\therefore \quad lt\,\frac{\Delta s}{\Delta x} = \frac{ds}{dx}. \tag{1}$$

But $$lt\,\frac{\text{chord } PP_1}{\Delta x} = lt \sec DPP_1 = \sec DPT = \frac{ds}{dx}. \qquad (2)$$

Dividing (1) by (2), $$\lim_{\Delta s \doteq 0}\left[\frac{\Delta s}{\text{chord } PP_1}\right] = 1. \qquad (3)$$

It follows from Art. 21 that in a limit an infinitesimal arc may be replaced by its chord.

EXERCISE II₂.

Differentiate by the general method.

1. $y = 2x + 3$.
2. $y = 3x + 7$.
3. $y = x^2 + 7x$.
4. $s = 3t^2 + 2t$.
5. $s = \dfrac{1}{3t}$.
6. $y = \dfrac{1}{x^2}$.
7. $y = 2t^2 - t$.
8. $y = 2t - t^3$.
9. $y = 2t^3 + 1$.
10. $y = 2t^2 - t^3$.
11. $y = \dfrac{3}{x+1}$.
12. $y = \dfrac{2}{1-t}$.
13. $y = \dfrac{2}{1-t^2}$.
14. $s = \dfrac{3}{1+t^2}$.
15. $y = \dfrac{4}{3-2x}$.
16. $y = \dfrac{5-2x}{3-x}$.
17. $y = \dfrac{2x+1}{x^2}$.
18. $y = \dfrac{3}{x^2-1}$.
19. $y = \dfrac{x^2}{2x+1}$.
20. $x = \dfrac{2y^2}{1+2y}$.

21. Suppose that the air contained in a cylinder has an initial volume of 100 cu. in. and an initial pressure of 30 lbs. per square inch. Assuming that the air expands according to the law expressed by pV = constant, compute the value of $\dfrac{\Delta p}{\Delta V}$ for $\Delta V = 2, 1, 0.1, 0.01$. Show that as $\Delta V \doteq 0$, the ratio of the increments has -0.3 for a limit.

22. Let $y = \sin\theta$. Take $\theta = 30°$, and let $\Delta\theta$ have the values, 1°, 30′, 5′. Make a table of the values of $\sin(\theta + \Delta\theta)$, Δy, and $\dfrac{\Delta y}{\Delta\theta}$. See if the value of the ratio $\dfrac{\Delta y}{\Delta\theta}$ seems to approach a limit, as $\Delta\theta$ is decreased towards zero.

ALGEBRAIC FUNCTIONS.

23. Formulas and Rules for Differentiation. — By the general method any function can be differentiated, but it is usually more directly done by formulas or rules established by the general method or by other methods.

In the following formulas u, v, y, and z denote variable quantities, functions of x; and a, c, and n, constant quantities. If in the formulas "$\frac{d}{dx}$" or "D_x" be substituted for "d," and in the rules "derivative" be substituted for differential, they are still valid.

[I] **If $y = x$, $dy = dx$.**

The differentials of equals are equal.

[II] $$d(a) \equiv 0.$$

The differential of a constant is zero.

[III] $$d(v + y + \cdot \cdot \cdot - z + c) = dv + dy + \cdot \cdot \cdot - dz.$$

The differential of a polynomial is the sum of the differentials of its terms.

[IV] $$d(ax) = a\,dx.$$

The differential of the product of a constant and a variable is the product of the constant and the differential of the variable.

[V_a] $$d(uy) = y\,du + u\,dy.$$

The differential of the product of two variables is the sum of the products of each variable by the differential of the other.

[V_b] $$d(uyz \ldots) = (yz \ldots)\,du + (uz \ldots)\,dy + (uy \ldots)\,dz + \cdots.$$

The differential of the product of any number of variables is the sum of the products of the differential of each by all the rest.

[VI] $$d\left(\frac{N}{D}\right) = \frac{D \cdot dN - N \cdot dD}{D^2}.$$

The differential of a fraction is the denominator by the differential of the numerator minus the numerator by the differential of the denominator, divided by the square of the denominator.

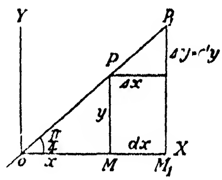
[VII] $$d(x^n) = nx^{n-1}\,dx.$$

The differential of a variable with a constant exponent is the product of the exponent and the variable with the exponent less one by the differential of the variable.

24. Derivation of [I]. — If y is continuously equal to x, it is evident that y and x must change at equal rates; that is,

$$\frac{dy}{dx} = \frac{dx}{dx}, \quad \therefore \quad dy = dx.$$

Since $\frac{dx}{dx} = 1$, the rate of x is the unit rate, so in general the rate of a variable with respect to itself is unity, or the derivative of $f(x)$, when $f(x)$ is x, is one.



Geometrically the locus of $y = x$ is the straight line through origin making angle $\phi = \frac{\pi}{4}$ with x-axis.

$$\tan\phi = \frac{y}{x} = \frac{\Delta y}{\Delta x} = 1,$$

and since $\frac{\Delta y}{\Delta x}$ is constant, $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 1, \quad \therefore \quad dy = dx.$

For examples of [I], if $y^2 = 2\,px$, $d(y^2) = d(2\,px)$; or if $x^2 + y^2 = a^2$, $d(x^2 + y^2) = d(a^2) = 0$.

25. Derivation of [II]. — By definition the value of a constant is fixed, therefore the rate of a constant is zero; that is,

$$\frac{da}{dx} = 0, \quad \therefore \quad da = 0.$$

If $y = a$, a change in x makes no change in y, hence $\Delta y = 0$, $\therefore \frac{\Delta y}{\Delta x} = 0$, and since $\frac{\Delta y}{\Delta x}$ is constant,

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = 0, \quad \therefore \quad dy = da = 0.$$

Geometrically the slope of $y = a$ (a line parallel to x-axis) is at every point zero.

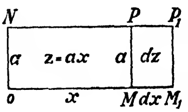
26. Derivation of [III]. — It is manifest that the rate of the sum of $v + y + \cdots - z + c$ is equal to the sum of the rates of its parts, $v, y, \ldots -z$ and c; that is,

$$\frac{d(v + y + \cdots - z + c)}{dx} = \frac{dv}{dx} + \frac{dy}{dx} + \cdots - \frac{dz}{dx} + \frac{dc}{dx}.$$

Multiplying by dx, since $dc = 0$, the result is [III]. The rule shows that differentials are summed like any other algebraic quantities. For example,

$$d(b^2x^2 \pm a^2y^2 - a^2b^2) = d(b^2x^2) \pm d(a^2y^2) - d(a^2b^2).$$

27. Derivation of [IV]. — Since $\Delta(ax) = a\Delta x$, the ratio of the increments is constant and ax changes uniformly with respect to x. Hence by definition of differentials $d(ax) = a\,dx$. If $y = ax$, $\frac{dy}{dx} = a$, slope of line. Geometrically, if $z = ax$ be area of a rectangle of base x and altitude a then the rectangle MPP_1M_1 is the change of z made by a change $\Delta x\ (= dx)$ of x, and being a uniform change is the differential of z,



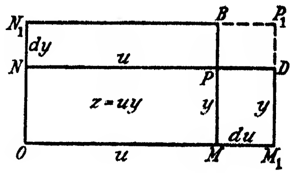
$$\therefore \quad dz = d(ax) = a\,dx.$$

For examples:

$$d(2px) = 2p\,dx, \quad \text{and} \quad d\left(\frac{x}{a}\right) = d\left(\frac{1}{a}x\right) = \frac{dx}{a}.$$

[V_a] will be seen to include [IV] as a special case.

28. Derivation of $[V_a]$. — Let $z = uy$; then z, a function of u is a function of y also. Geometrically, let u and y be the base and altitude of a variable rectangle conceived as generated by the side y moving to the right and the upper base u moving upward; then $z = uy$ is the area. If at the value $OMPN$, $du = MM_1$, and $dy = NN_1$, the differential of the area is $MM_1DP + NPBN_1$, as that sum *would* be the change of the area of the rectangle due to the change of u and y, if at the value $OMPN$ the change of its area became uniform. Hence $dz = d(uy) = y\,du + u\,dy$. Here $\Delta z = \Delta(uy) = d(uy) + PDP_1B$, since that sum is the actual change of the area due to the change of u and y.



It is to be noted that if $y = u$, then the rectangle is a square and area $z = u^2$,

$$\therefore \quad d(u^2) = u\,du + u\,du = 2\,u\,du.$$

If $y = a$, $z = au$, $dz = a\,du + u\,da = a\,du$, since $da = 0$. Hence [IV] is a special case of $[V_a]$.

29. Derivation of $[V_b]$. — To prove $d(uyz) = yz\,du + uz\,dy + uy\,dz$. If in $[V_a]$, yz is put for y;

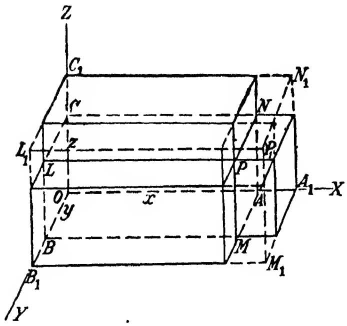
$$\begin{aligned} d(uyz) &= yz\,du + u d(yz) \\ &= yz\,du + u(z\,dy + y\,dz) \\ &= yz\,du + uz\,dy + uy\,dz. \end{aligned}$$

By repeating this process the rule is proved for any number of variables. If $y = z = u$,

then $\quad d(uyz) = d(u^3) = u^2\,du + u^2\,du + u^2\,du = 3\,u^2\,du.$

To derive $d(uyz)$ geometrically, let $V = xyz = uyz$.

If x, y, and z be the edges of a variable right parallelopiped conceived as generated by the face yz moving to the right, the face xz moving to the front, and the face xy moving



upward, then the volume is the product of the three edges; that is, $V = xyz$.

If at the value OP, $dx = AA_1$, $dy = BB_1$, and $dz = CC_1$, the differential of the volume is $PA_1 + PB_1 + PC_1$; as that sum *would* be the change of the volume of the parallelopiped due to the change of x, y, and z, if at the value OP the change of its volume became uniform. Hence

$$dV = d(xyz) = yz\,dx + xz\,dy + xy\,dz.$$

Here

$$\Delta V = dV + PN_1 + PL_1 + PM_1 + PP_1,$$

since that sum is the actual change of the volume due to the change of x, y, and z. If $y = z = x$, then the parallelopiped is a cube and $V = x^3$,

$$\therefore \quad d(x^3) = x^2\,dx + x^2\,dx + x^2\,dx = 3\,x^2\,dx.$$

30. Derivation of [VI]. — Let $z = \dfrac{y}{x}$ (x and y independent), then $zx = y$.

$$\therefore \quad d(zx) = x\,dz + z\,dx = dy, \text{ by } [\mathbf{V}_a].$$

Solving,

$$dz = \frac{dy}{x} - \frac{z\,dx}{x},$$

$$\therefore \quad d\left(\frac{y}{x}\right) = \frac{dy}{x} - \frac{\left(\dfrac{y}{x}\right)dx}{x} = \frac{x\,dy - y\,dx}{x^2}.$$

Corollary. —

$$d\left(\frac{a}{x}\right) = -\frac{a\,dx}{x^2};$$

for $$d\left(\frac{a}{x}\right) = \frac{x\,da - a\,dx}{x^2} = -\frac{a\,dx}{x^2}, \quad \text{since } da = 0.$$

$$d\left(\frac{x}{a}\right) = \frac{dx}{a};$$

for $\quad d\left(\frac{x}{a}\right) = \frac{a\,dx - x\,da}{a^2} = \frac{dx}{a},\quad$ since $da = 0$;

hence, for a fraction with constant denominator, use [IV]. For another derivation of [VI], see *Corollary* of next Art. 31.

31. Derivation of [VII]. — I. *When the exponent is a positive integer.*

(*a*) If n is a positive integer, $x^n = x \cdot x \cdot x \cdot$ to n factors; hence,

$$\begin{aligned} d(x^n) &= d(x \cdot x \cdot x \text{ to } n \text{ factors}) \\ &= x^{n-1}\,dx + x^{n-1}\,dx + x^{n-1}\,dx + \cdots \\ &\qquad \text{to } n \text{ terms, by } [\mathrm{V}_b], \\ &= nx^{n-1}\,dx. \end{aligned}$$

(*b*) *By the general method.* Let $y = x^n$.

$$\left.\begin{aligned} y + \Delta y &= (x + \Delta x)^n = x^n + nx^{n-1}\,\Delta x + (\text{terms} \\ &\qquad \text{with common factor } \overline{\Delta x}^2), \\ \Delta y &= nx^{n-1}\,\Delta x + (\text{terms with factor } \overline{\Delta x}^2) \\ \frac{\Delta y}{\Delta x} &= nx^{n-1} + (\text{terms with factor } \Delta x), \end{aligned}\right\}\ \begin{array}{l}\text{by}\\ \text{Binomial}\\ \text{Theorem.}\end{array}$$

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = nx^{n-1}, \quad \therefore \quad dy = nx^{n-1}\,dx.$$

II. *When the exponent is a positive fraction.*

$$\begin{aligned} \text{Let} \qquad y &= x^{\frac{m}{n}}, \\ \text{then} \qquad y^n &= x^m, \\ \therefore \quad d(y^n) &= d(x^m), \\ ny^{n-1}\,dy &= mx^{m-1}\,dx, \\ \therefore \quad dy &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}}\,dx = \frac{m}{n}\frac{x^{m-1}\,y}{y^n}, \end{aligned}$$

$$\therefore \quad d\left(x^{\frac{m}{n}}\right) = \frac{m}{n}\frac{x^{m-1}x^{\frac{m}{n}}}{x^m}dx = \frac{m}{n}x^{\frac{m}{n}-1}dx.$$

III. *When the exponent is negative.*

Let $y = x^{-n}$, n being integral or fractional; then $y = \dfrac{1}{x^n}$;

$$\therefore \quad dy = d\left(\frac{1}{x^n}\right) = -\frac{nx^{n-1}}{x^{2n}}dx, \text{ by [VI], } Cor.\text{, Art. 30,}$$

$$\therefore \quad dy = d\,(x^{-n}) = -nx^{-n-1}\,dx.$$

Corollary. — $d\left(\dfrac{x}{y}\right) = d\,(xy^{-1}) = y^{-1}\,dx - xy^{-2}\,dy$

$$= \frac{dx}{y} - \frac{x\,dy}{y^2} = \frac{y\,dx - x\,dy}{y^2}. \qquad \text{[VI]}$$

Note. — A general proof of [VII] by logarithms, given further on (Art. 37), includes the case where the exponent is incommensurable. So the Formula or Rule is valid for *any* constant exponent. It is called the *Power Formula* and is of most frequent application.

Examples. —

$$d\,(\sqrt{x}) = d\,(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{dx}{2\sqrt{x}}.$$

$$d\left(\frac{1}{\sqrt{x}}\right) = d\,(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}}\,dx = -\frac{dx}{2\sqrt{x^3}}.$$

$$d\,(x^{\sqrt{2}}) = \sqrt{2}\,x^{\sqrt{2}-1}\,dx \;(= 1.414\,x^{.414}\,dx, \text{ approximately}).$$

$$d\,(x^{\pi}) = \pi x^{\pi-1}\,dx \;(= 3.1416\,x^{2.1416}\,dx, \text{ approximately}).$$

$$d\,((ax+b)^n) = n\,(ax+b)^{n-1}\,d\,(ax+b) = na\,(ax+b)^{n-1}\,dx,$$

$$\therefore \quad \frac{d}{dx}((ax+b)^n) = na\,(ax+b)^{n-1}.$$

Note. — The last example may be seen to be an application of the formula for the derivative of a function of a function. For let $y = (ax+b)^n$ and put $z = ax + b$, then $y = z^n$; now y is a function of z, and z is a function of x; that is, y is

a function of a function of x. The formula given in *Remarks*, Art. 19, is $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$;

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}((ax+b)^n) = \frac{d(z^n)}{dz} \cdot \frac{d(ax+b)}{dx} = nz^{n-1} \cdot a$$
$$= na(ax+b)^{n-1}.$$

In applying Rule [VII], if all within the parenthesis, as $(ax + b)$, is regarded as the variable, the actual substitution of z may be dispensed with in getting the derivative of such functions.

EXERCISE III.

By one or more of the formulas I–VII differentiate:

1. $y = 6\sqrt[3]{x^2} + \dfrac{4}{\sqrt{x^3}} - \dfrac{2}{x} + \dfrac{3}{x^4}$.

$$dy = d(6x^{\frac{2}{3}}) + d(4x^{-\frac{3}{2}}) - d(2x^{-1}) + d(3x^{-4}).$$
$$\frac{dy}{dx} = \frac{4}{\sqrt[3]{x}} - \frac{6}{\sqrt{x^5}} + \frac{2}{x^2} - \frac{12}{x^5}.$$

2. $y = 3x^3 - 4x^2 - 2$.

$$dy = d(3x^3) - d(4x^2) - d(2).$$
$$dy = 9x^2\,dx - 8x\,dx - 0;$$
$$\frac{dy}{dx} = 9x^2 - 8x = (9x - 8)x;$$

that is, y changes $(9x - 8)x$ times as much as x.

When $x = -1$, y is increasing at the rate of 17 to 1 of x;
$x = \frac{8}{9}$, y is neither increasing nor decreasing;
$x = 0$, y is neither increasing nor decreasing;
$x = \frac{1}{9}$, y is decreasing at the rate of $\frac{7}{9}$ to 1 of x;
$x = 1$, y is changing at the same rate as x;
$x = -\frac{1}{9}$, y is changing at the same rate as x;
$x = 2$, y is increasing at the rate of 20 to 1 of x.

Note. — In this way the meaning of each differential equation may be shown.

3. $y = (1 + 2x^2)(1 + 4x^3)$. $\qquad dy = 4x(1 + 3x + 10x^3)\,dx$.

$$dy = (1 + 2x^2)\,d(1 + 4x^3) + (1 + 4x^3)\,d(1 + 2x^2);$$

or $\qquad dy = d(1 + 2x^2 + 4x^3 + 8x^5)$.

4. $y = (x+1)^5(2x-1)^3.$ $\quad dy = (16x+1)(x+1)^4(2x-1)^2\,dx.$

5. $y = (a+x)\sqrt{a-x}.$ $\quad \frac{dy}{dx} = \frac{a-3x}{2\sqrt{a-x}}.$

6. $y=(1-3x^2+6x^4)(1+x^2)^3.$ $\quad dy = 60x^5(1+x^2)^2\,dx.$

7. $y = (x^{\frac{1}{3}} - a^{\frac{1}{3}})^4.$ $\quad \frac{dy}{dx} = \frac{4(x^{\frac{1}{3}}-a^{\frac{1}{3}})^3}{3x^{\frac{2}{3}}}.$

8. $y = \frac{x+a^2}{x+b}.$ $\quad \frac{dy}{dx} = \frac{b-a^2}{(x+b)^2}.$

$$dy = \frac{(x+b)\,d(x+a^2) - (x+a^2)\,d(x+b)}{(x+b)^2}.$$

9. $y = \frac{2x-1}{(x-1)^2}.$ $\quad \frac{dy}{dx} = -\frac{2x}{(x-1)^3}.$

10. $y = x(x^3+5)^{\frac{4}{3}}.$ $\quad \frac{dy}{dx} = 5(x^3+1)(x^3+5)^{\frac{1}{3}}.$

11. $y = \frac{2x^4}{a^2-x^2}.$ $\quad \frac{dy}{dx} = \frac{8a^2x^3-4x^5}{(a^2-x^2)^2}.$

12. $y = \sqrt{ax^2+bx+c}.$ $\quad \frac{dy}{dx} = \frac{2ax+b}{2\sqrt{ax^2+bx+c}}.$

13. $y = \sqrt{\frac{1+x}{1-x}}.$ $\quad \frac{dy}{dx} = \frac{1}{(1-x)\sqrt{1-x^2}}.$

14. $y = \frac{x}{\sqrt{a^2-x^2}}.$ $\quad \frac{dy}{dx} = \frac{a^2}{\sqrt{(a^2-x^2)^3}}.$

15. $y = \frac{x^n}{(1+x)^n}.$ $\quad \frac{dy}{dx} = \frac{nx^{n-1}}{(1+x)^{n+1}}.$

16. $y = \frac{x^n+1}{x^n-1}.$ $\quad \frac{dx}{dy} = -\frac{2nx^{n-1}}{(x^n-1)^2}.$

17. $y = \frac{x}{\sqrt{a^2+x^2}-x}.$ $\quad \frac{dy}{dx} = \frac{1}{a^2}\left[\frac{a^2+2x^2}{\sqrt{a^2+x^2}}+2x\right].$

Rationalize the denominator before differentiating.

18. $x = t(t^2+a^2)^{\frac{n-1}{2}}.$ $\quad \frac{dx}{dt} = (nt^2+a^2)(t^2+a^2)^{\frac{n-3}{2}}.$

19. A vessel is sailing due north 20 miles per hour. Another vessel, 40 miles north of the first, is sailing due east 15 miles per hour. At what rate are they approaching each other after one hour? After 2 hours? *Ans.* Approaching 7 mi. per hr.; separating 15 mi. per hr. When will they cease to approach each other, and what is then their distance apart? *Ans.* After 1 hr. 16 m. 48 sec.; 24 mi.

20. If a body moves so that $s = \sqrt{t}$, show that the acceleration is negative and proportional to the cube of the velocity. Negative sign shows what?

21. If $x = at$ and $y = bt - \frac{1}{2}ct^2$, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$.

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{b - ct}{a}. \qquad \frac{dx}{dy} = \frac{dx}{dt} \Big/ \frac{dy}{dt} = \frac{a}{b - ct}. \quad \text{(By Art. 19}(a).)$$

Or $$y = \frac{bx}{a} - \frac{1}{2}\frac{cx^2}{a^2}; \quad \therefore \quad \frac{dy}{dx} = \frac{b}{a} - \frac{cx}{a^2} = \frac{b}{a} - \frac{cat}{a^2} = \frac{b - ct}{a}.$$

22. If $\rho = \sqrt{t}$ and $\theta = t^2 - 10$, find $\frac{d\rho}{d\theta}$.

$$\frac{d\rho}{d\theta} = \frac{d\rho}{dt} \Big/ \frac{d\theta}{dt} = \frac{1}{2t^{\frac{1}{2}}} \Big/ 2t = \frac{1}{4t^{\frac{3}{2}}}. \quad \text{(By Art. 19, Remarks.)}$$

Or $$\rho^2 = t = (\theta + 10)^{\frac{1}{2}}. \quad \therefore \quad 2\rho\, d\rho = \frac{d\theta}{2(\theta + 10)^{\frac{1}{2}}} = \frac{1}{4t^{\frac{3}{2}}}.$$

23. The equation $pV = C$ expresses Boyle's law, C being a constant. Find $\frac{dp}{dV}$ and $\frac{dV}{dp}$. (See Ex. 5, Art. 20.)

24. The heat H required to raise a unit weight of water from 0° C. to a temperature T is given by the equation,

$$H = T + 0.00002\,T^2 + 0.0000003\,T^3.$$

(*a*) Find $\frac{dH}{dT}$. (*b*) Compute the numerical value of the rate for $T = 35°$. *Ans.* (*b*) 1.0025025.

25. A vessel in the form of an inverted circular cone of semi-vertical angle 30°, is being filled with water at the uniform rate of one cubic foot per minute. At what rate is the surface of the water rising when the depth is 6 inches? When 1 foot? When 2 feet?

Ans. 0.76 in.; 0.19 in.; 0.05 in., per sec.

26. Show that the slope of the tangent to the curve $y = x^3 + 4$ is never negative. Find the slope for $x = 0$, for $x = 2$. For what values of x does the slope decrease as x increases?

LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

32. Formulas and Rules for Differentiation.—

$$[\text{VIII}_a] \quad d(\log_b x) = \frac{m}{x}\,dx \ (x \text{ positive}).$$

$$(m = \log_b e = 0.434 \ldots, \text{ for } b = 10).$$

$$[\text{VIII}_b] \quad d(\log_e x) = \frac{1}{x}\,dx \ (x \text{ positive}).$$

$$(m = \log_e e = 1,\ e = 2.718 \ldots).$$

The differential of the logarithm of a variable is the product of the modulus of the system and the reciprocal of the variable by the differential of the variable.

$$[\mathrm{IX}_a] \qquad \boldsymbol{d\,(b^x) = b^x \log_e b\, dx} \ (b \text{ positive}).$$

$$[\mathrm{IX}_b] \qquad \boldsymbol{d\,(e^x) = e^x\, dx}.$$

The differential of an exponential function with constant base and variable exponent is the product of the function and the Napierian logarithm of the base by the differential of the exponent.

$$[\mathrm{X}] \qquad \boldsymbol{d\,(y^x) = y^x \log_e y\, dx + xy^{x-1}\, dy},$$

(y *positive* and independent of x).

The differential of an exponential function with base and exponent variable is the sum of the results obtained by differentiating as though the base were constant and then as though the exponent were constant.

33. Derivation of [VIII$_a$] and [VIII$_b$]. —

(i) Taking n an arbitrary constant, let

$$x = ny. \tag{1}$$

$$\log_b x = \log_b (ny) = \log_b y + \log_b n. \tag{2}$$

$$d\,(\log_b x) = d\,(\log_b y)\,[+ d\,(\log_b n) = 0]. \tag{3}$$

Differentiating (1) and dividing result by (1),

$$\frac{dx}{x} = \frac{dy}{y}. \tag{4}$$

Dividing (3) by (4) gives as result,

$$d\,(\log_b x) \Big/ \frac{dx}{x} = d\,(\log_b y) \Big/ \frac{dy}{y}. \tag{5}$$

It is manifest that the equal ratios in (5) are constant for any particular value of y. Let m denote the constant value of the ratio when $y = y_1$; then

$$d\,(\log_b x) = \frac{m}{x}\, dx, \tag{6}$$

when $x = ny_1$; and, as n is an arbitrary constant, ny_1 denotes any positive number. Hence (6) or [VIII$_a$] holds true for all positive values of x, m being a constant. The constant m is called the modulus of the system of logarithms, whose base is denoted by b in this derivation. The general base is often denoted by a.

The system whose modulus is unity is called the Napierian or natural system. The symbol for the base of this system is e, called the Napierian base from the name of the discoverer of logarithms.

Hence $$d(\log_e x) = \frac{1}{x}dx. \quad \text{[VIII}_b\text{]}$$

(ii) By the general method of limits. (x positive.) Let

$$y = \log_b x. \quad y + \Delta y = \log_b (x + \Delta x),$$

$$\Delta y = \log_b (x + \Delta x) - \log_b x = \log_b \left(1 + \frac{\Delta x}{x}\right),$$

$$\therefore \quad b^{\Delta y} = \left(1 + \frac{\Delta x}{x}\right).$$

Raising each member to $\frac{x}{\Delta x}$ power gives

$$b^{x\frac{\Delta y}{\Delta x}} = \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

The limit of each member as $\Delta x \doteq 0$ gives

$$\lim_{\Delta x \doteq 0} b^{x\frac{\Delta y}{\Delta x}} = \lim_{\Delta x \doteq 0} \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \lim_{n=\infty} \left(1 + \frac{1}{n}\right)^n, \text{ putting } \frac{\Delta x}{x} = \frac{1}{n},$$

so that (if x is not zero), as $\Delta x \doteq 0$, $n = \infty$;

$$\lim_{\Delta x \doteq 0} b^{x\frac{\Delta y}{\Delta x}} = b^{x\frac{dy}{dx}} = \lim_{n=\infty} \left(1 + \frac{1}{n}\right)^n = e$$

(denoting the limit by that letter);

$$\therefore \quad x\frac{dy}{dx} = \log_b e, \quad \text{or } \frac{dy}{dx} = \frac{\log_b e}{x} = \frac{m}{x}$$

($m = \log_b e =$ the modulus);

$$\therefore \quad dy = d(\log_b x) = \frac{m}{x}dx. \qquad [\mathrm{VIII}_a]$$

Hence,

$$d(\log_e x) = \frac{1}{x}dx, \text{ since } \log_e e = 1 = \text{the modulus.} \qquad [\mathrm{VIII}_b]$$

The limit of $\left(1+\frac{1}{n}\right)^n$ as n is increased without limit is e, the Napierian base. (See next Art. 34.)

34. $\lim\limits_{n=\infty}\left(1+\frac{1}{n}\right)^n = e.$ (See Ex. 7, Art. 162.)

When n is a positive integer, by the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3}+\cdots$$

$$= 1+1+\frac{1\left(1-\frac{1}{n}\right)}{1\cdot 2}+\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\cdot 2\cdot 3}+\cdots\cdot$$

In the expansion there are $(n+1)$ terms in all, and every term after the second can be written in the form given to the 3rd and 4th terms. As $n=\infty$, $\frac{1}{n} \doteq 0$,

$$\therefore \quad \lim_{n\doteq\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots$$
$$= e = 2.7182818\ldots.$$

Note 1. — The limit is denoted by e, which is an irrational number, and was proved by Hermite, in 1874, to be transcendental or non-algebraic. The number e was the first number to be proved transcendental. Not until 1882 was the attempt to prove the number π transcendental successful. This was finally done by Lindemann. The proofs consist in showing that neither of the two numbers is the root of an algebraic equation with integers for coefficients. Algebraic real numbers are defined as those real numbers which are roots of such an equation. The importance of these two numbers, considered the most important in mathematics,

warrants some notice. They are connected by the remarkable relation, $e^{\pi\sqrt{-1}} = 1$. (See Ex. 10, Exercise XXX.)

Note 2. — The above derivation of the limit is not complete, for the result is true not only when n is "a positive integer," but also for n positive or negative, integral, fractional, or incommensurable. The value of the limit, e, can be easily computed to any desired degree of precision by taking a sufficient number of terms of the series. Twelve terms gives the result correct to seven decimals; that is, $e = 2.7182818 \ldots$

By comparing the sum of $(n + 1)$ terms of the series with the sum, $1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{n-1}}$, which is greater than the other, and equal to $3 - \frac{1}{2}^{n-1}$, it is manifest that no matter how great n may be, the sum of the $(n + 1)$ terms is certainly finite and less than 3. The

$$\lim_{n \doteq \infty}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)$$

may be considered as e, or as usually written,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

to infinity. (See Ex. 5, Art. 154.)

Without expanding, the $\lim\limits_{n \doteq \infty}\left(1 + \frac{1}{n}\right)^n$ can be computed to any desired number of decimals by giving increasing values to n; thus,

$$\begin{aligned}(1 + \tfrac{1}{10})^{10} &= 2.59374.\\ (1.01)^{100} &= 2.70481.\\ (1.001)^{1000} &= 2.71692.\\ \ldots &= \ldots\\ (1.000001)^{1\,000\,000} &= 2.71828.\end{aligned}$$

The last number agrees with the value of e, the required limit, to five decimals.

Corollary. — $\operatorname*{Lim}\limits_{n \doteq 0} (1 + n)^{\frac{1}{n}} = e$. (See Ex. 8, Art. 162.)

35. Derivation of [IX_a] and [IX_b]. —

(i) Let $y = b^x$, then $\log_e y = x \log_e b$.

$$d(\log_e y) = d(x \log_e b), \quad \text{or} \quad \frac{dy}{y} = \log_e b\, dx,$$

$$\therefore \quad dy = d(b^x) = b^x \log_e b\, dx \text{ (b being positive).} \quad [\mathrm{IX}_a]$$

Hence, $d(e^x) = e^x dx$ (since $\log_e e = 1$). [IX_b]

(ii) If $y = b^x$, $x = \log_b y$. $dx = d(\log_b y) = \dfrac{\log_b e\, dy}{y}$.

$$\therefore \quad dy = \frac{y}{\log_b e} dx = b^x \log_e b\, dx \left(\text{since } \frac{1}{\log_b e} = \log_e b\right). \quad [\mathrm{IX}_a]$$

Hence, $$d(e^x) = e^x dx. \quad [\mathrm{IX}_b]$$

(iii) By the general method of limits. Let

$$y = e^x. \qquad y + \Delta y = e^{x+\Delta x}.$$

$$\Delta y = e^{x+\Delta x} - e^x = e^x(e^{\Delta x} - 1).$$

$$\frac{\Delta y}{\Delta x} = \frac{e^x(e^{\Delta x} - 1)}{\Delta x}.$$

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx} = \lim_{\Delta x \doteq 0}\left[e^x\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)\right] = e^x \lim_{\Delta x \doteq 0}\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)$$

$$= e^x \left(\text{since } \lim_{\Delta x \doteq 0}\left(\frac{e^{\Delta x} - 1}{\Delta x}\right) = 1\right) \quad (\textit{Cor.}, \text{Art. } 36);$$

$$\therefore \quad dy = d(e^x) = e^x dx. \quad [\mathrm{IX}_b]$$

Corollary.— $d(b^x) = b^x \log_e b\, dx$. [$\mathrm{IX}_a$]

For if $\dfrac{1}{m} = \log_e b$, $b^x = e^{\frac{x}{m}}$, since $b = e^{\frac{1}{m}}$;

$$d(b^x) = d\left(e^{\frac{x}{m}}\right);$$

$$\therefore \quad d(b^x) = e^{\frac{x}{m}} \frac{dx}{m} = b^x \frac{dx}{m};$$

$$\therefore \quad d(b^x) = b^x \log_e b\, dx.$$

36. $\displaystyle \lim_{n=\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$

In $\lim\limits_{n=\infty}\left(1+\frac{x}{n}\right)^n$, if $x \neq 0$, by putting $n = Nx$, when $n = \infty$ so is $N = \infty$; hence

$$\left(1+\frac{x}{n}\right)^n = \left(1+\frac{1}{N}\right)^{Nx} = \left\{\left(1+\frac{1}{N}\right)^N\right\}^x,$$

and

$$\lim_{n=\infty}\left(1+\frac{x}{n}\right)^n = \lim_{N=\infty}\left\{\left(1+\frac{1}{N}\right)^N\right\}^x = \left\{\lim_{N=\infty}\left(1+\frac{1}{N}\right)^N\right\}^x = e^x,$$

since $\lim f(x) = f(\lim x)$, the case of a function of a function. (See *Remarks*, Art. 19.)

By exactly the same method as in Art. 34, it may be shown that

$$e^x = \lim_{n=\infty}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right),$$

by expanding $\left(1+\frac{x}{n}\right)^n$ for positive integral values of n.

It can be shown that the limit of this series is a finite number for all finite values of x no matter how great n may be. (See Art. 152 and Ex. 5, Art. 154.)

Corollary. — $\operatorname{Lim}\limits_{x \doteq 0}\left(\frac{e^x - 1}{x}\right) = 1$, which may be put in the form,

$$\lim_{\Delta x \doteq 0}\left(\frac{e^{\Delta x} - 1}{\Delta x}\right) = 1.$$

37. Derivation of [X]. —

Let $z = y^x$, then $\log_e z = x \log_e y$. (y positive and independent of x.)

$$d(\log_e z) = d(x \log_e y),$$

$$\frac{dz}{z} = \log_e y \, dx + x\frac{dy}{y};$$

$$\therefore \quad dz = d(y^x) = y^x \log_e y \, dx + xy^{x-1}\, dy \ (y \text{ positive}). \quad [X]$$

Note. — Formulas [VII], [IX$_a$], [IX$_b$] are seen to result from [X] as special cases.

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Let $y = x^n$, then $\log_e y = n \log_e x$,

$$d(\log_e y) = d(n \log_e x),$$

$$\frac{dy}{y} = n\frac{dx}{x};$$

$$\therefore \quad dy = d(x^n) = nx^{n-1}\,dx. \quad \text{[VII]}$$

If x were negative, to avoid logarithms of negative numbers, both members of $y = x^n$ are squared before differentiating.

This derivation of [VII] includes the case where n is incommensurable.

38. Modulus. — In Art. 33 (ii), it appears that $\log_b e$ is the modulus of the system of logarithms whose base is b. Hence, when the base is 10, as in the common system, and the value of e is known, a table of logarithms will give the value of the modulus of the common system to as many decimals as the table gives. The modulus of the common system, denoted by M, is $\log_{10} e = 0.43429 \ldots$ If this value of M is deduced independently of any knowledge of the value of e, which can be done; then the value of e can be gotten from a table of logarithms; for, since $M = \log_{10} e$, then $e = 10^M$; that is, e is the number whose common logarithm is 0.43429. . . .

In Art. 35 (i) and (ii), it appears that $\log_e b$ is equal to

$$\frac{1}{\log_b e}, \quad \therefore \quad \log_e 10 = \frac{1}{\log_{10} e} = \frac{1}{.434\ldots} = 2.3026$$

approximately. (See Ex. 6, Art. 154.)

To get these results independently, let x be any number whose logarithm in the system with base 10 is l, and in that with base e is l'; then $10^l = x$ and $e^{l'} = x$;

$$\therefore \quad 10^l = e^{l'}. \tag{1}$$

Let
$$10^M = e; \tag{2}$$

$$\therefore \quad 10^l = 10^{Ml'}; \quad \therefore \quad l = Ml' \text{ or } \frac{l}{l'} = M, \tag{3}$$

and since 10 and e are constant, so also is M. From (2), $M = \log_{10} e$, or from (1) $l = \log_{10} e^{l'} = l' \log_{10} e$;

$$\therefore \quad \frac{l}{l'} = \log_{10} e = M; \text{ or in general, } \log_b e = m.$$

Since $l = Ml'$, or $\log_{10} x = M \log_e x$, it follows that the common logarithm of any number is equal to M times the Napierian logarithm of that number.

Now $\quad d(\log_{10} x) = Md(\log_e x)$ or $\dfrac{d(\log_{10} x)}{d(\log_e x)} = M$,

or m for base b, and dividing [VIII_a] by [VIII_b] gives

$$\frac{d(\log_b x)}{d(\log_e x)} = m, \text{ the modulus;}$$

$$\therefore \quad M = \log_{10} e = \text{modulus of common system } (b = 10),$$

and $m = \log_e e = 1$, modulus of natural system ($b = e$).

From (1) above, $l \log_e 10 = l'$ or $\dfrac{l}{l'} = \dfrac{1}{\log_e 10} = M$, by (3);

$$\therefore \quad \log_{10} e = \frac{1}{\log_e 10}, \text{ or } \log_e 10 = \frac{1}{M} = \frac{1}{0.434\ldots} = 2.3026,$$

approximately.

To summarize in two equations:

Common log = 0.434 times natural log.
Natural log = 2.3026 times common log.

Note. — Since the modulus of the natural system is unity the differentials of logarithms are simpler when the logarithms are in that system; hence, in the Calculus and in most analytic work, Napierian logarithms are employed for the most part. Any finite number except *one* could be made the base of a system of logarithms. For computation the common logarithms are the best, as having the base 10 affords rules for the integral part of the logarithms and obviates the necessity of that part appearing in the tables. It is usual in writing log for logarithm to omit the subscript

indicating the base, when no ambiguity results. Hereafter, when no subscript to log appears, e will be understood.

39. Logarithmic Differentiation. — Exponential functions and also those involving products and quotients are often more easily differentiated by first taking logarithms. This method which is used in the last two derivations (Art. 35 and Art. 37) is called *logarithmic differentiation.*

To derive $[\mathrm{V}_a]$, let $z = uy$, then $\log z = \log u + \log y$,

$$d(\log z) = \frac{dz}{z} = \frac{du}{u} + \frac{dy}{y} = d(\log u) + d(\log y);$$

$$\therefore \quad dz = d(uy) = y\,du + u\,dy.$$

To derive $[\mathrm{V}_b]$, let

$$V = uyz, \text{ then } \log V = \log u + \log y + \log z,$$

$$d(\log V) = \frac{dV}{V} = \frac{du}{u} + \frac{dy}{y} + \frac{dz}{z};$$

$$\therefore \quad dV = d(yz) = yz\,du + uz\,dy + uy\,dz.$$

To derive [VI], let $z = y/x$, then $\log z = \log y - \log x$,

$$d(\log z) = \frac{dz}{z} = \frac{dy}{y} - \frac{dx}{x} = d(\log y) - d(\log x);$$

$$\therefore \quad dz = d\left(\frac{y}{x}\right) = \frac{dy}{x} - \frac{y\,dx}{x^2} = \frac{x\,dy - y\,dx}{x^2}.$$

40. Relative Rate. Percentage Rate. — The logarithmic derivative of a function may be defined as the *relative* rate of increase of the function. Thus, when $y = f(x)$,

$$\frac{\frac{dy}{dx}}{y} = \frac{f'(x)}{f(x)} \text{ is the relative rate of } y.$$

Hence, when $z = xy$ and therefore $\log z = \log x + \log y$;

$$\frac{\frac{dz}{dx}}{z} = \frac{\frac{dx}{dx}}{x} + \frac{\frac{dy}{dx}}{y};$$

that is, *the relative rate of increase of a product is the sum of the relative rates of increase of the factors.* If the logarithmic derivative is multiplied 100 times, the product expresses the percentage rate of increase. Thus when

$$\frac{dy}{dx} = Ky, \quad 100\frac{\frac{dy}{dx}}{y} = 100\,K$$

is the percentage rate of increase, and is here constant.

EXERCISE IV.

By one or more of the formulas [I] to [X], differentiate:*

1. $y = \log_b x^3 = 3\log_b x.$ $\quad \frac{dy}{dx} = \frac{3\log_b e}{x} = \frac{3m}{x}.$

2. $y = (\log_{10} x)^3.$ $\quad \frac{dy}{dx} = \frac{3\log_{10} e}{x}(\log_{10} x)^2 = \frac{1.302\ldots}{x}(\log_{10} x)^2.$

3. $y = x\log x.$ $\quad \frac{dy}{dx} = \log x + 1.$

4. $y = \frac{1}{x\log x}.$ $\quad \frac{dy}{dx} = -\frac{1+\log x}{(x\log x)^2}.$

5. $y = \log\frac{ax-b}{ax+b} = \log(ax-b) - \log(ax+b).$ $\quad \frac{dy}{dx} = \frac{2ab}{a^2x^2-b^2}.$

6. $y = \log\frac{1+\sqrt{x}}{1-\sqrt{x}} = \log(1+\sqrt{x}) - \log(1-\sqrt{x}).$

7. $y = \log x\sqrt{1+x^2}.$ $\quad \frac{dy}{dx} = \frac{2x^2+1}{x^3+x}.$

8. $y = b^x e^x.$ $\quad \frac{dy}{dx} = (1+\log b)\,b^x e^x.$

9. $y = \log(a^x + b^x).$ $\quad \frac{dy}{dx} = \frac{a^x\log a + b^x\log b}{a^x+b^x}.$

10. $y = x^5\,5^x.$ $\quad \frac{dy}{dx} = x^x 5^4(5 + x\log 5).$

11. $y = x^x.$ $\quad \frac{dy}{dx} = x^x(\log x + 1).$

12. $y = x^{e^x}.$ $\quad \frac{dy}{dx} = x^{e^x}e^x\,\frac{1+x\log x}{x}.$

* In some of the examples logarithmic differentiation is employed to advantage; that is, take logarithms first and then differentiate.

Here $\log y = e^x \log x \quad \therefore \quad \frac{dy}{y} = e^x \frac{dx}{x} + \log x e^x \, dx.$

13. $y = x^{\log x}.$ $\qquad dy = 2\, x^{\log x - 1} \log x \cdot dx.$

14. $y = \log (\log x).$ $\qquad \frac{dy}{dx} = \frac{1}{x \log x}.$

15. $x^{\log a} = a^{\log x}.$ (Differentiate both members and verify results.)

16. $(x + e^x)^4 = x^4 + 4\, x^3 e^x + 6\, x^2 e^{2x} + 4\, x e^{3x} + e^{4x}.$ (Do as in 15.)

17. $(e^x + e^{-x})^2 = e^{2x} + 2 + e^{-2x}.$ (Do as in 15.)

18. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$ $\qquad \frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}.$

19. $y = \log \frac{e^x}{1 + e^x}.$ $\qquad \frac{dy}{dx} = \frac{1}{1 + e^x}.$

20. $y = (\log x)^x.$ $\qquad \frac{dy}{dx} = (\log x)^x \left(\frac{1}{\log x} + \log \log x\right).$

21. Find the slope of the curve $y = \log_{10} x$, or $x = 10^y$, showing that the results are identical. What is the value of the slope at (1, 0)? What is the slope of the curve $y = \log_e x$, or $x = e^y$, and its value at (1, 0)?

22. Find the slope of the curve $x = \log_e y$, or $y = e^x$; and note that the slope at any point has the value of the ordinate at that point. Value of the slope at $x = 0$? At $x = 1$? At $x = -\infty$?

23. Find the slope of the curve $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ at $x = 0$.

Ans. 0.

What is the abscissa of the point where the curve is inclined 45° to the x-axis?

Ans. $x = a \log_e (1 + \sqrt{2}).$

24.* Find the value of x when $\log_{10} x$ increases at the same rate as x.

Ans. $x = \log_{10} e = 0.4343 \ldots$

* Since
$$d (\log_{10} x) = \log_{10} e \cdot \frac{dx}{x};$$
$$dx = x \cdot \frac{d (\log_{10} x)}{\log_{10} e} = 2.3026\, x \cdot d (\log_{10} x);$$
hence, any number N increases about 2.3 N times as fast as $\log_{10} N$. When
$$N = 0.4343 \ldots, \; dN = 0.4343 \times 2.3026\, d (\log_{10} N) = d (\log_{10} N).$$
Find how much faster x is increasing than $\log_{10} x$ for $x = 1$.

Ans. $2.3026\, x = 2.3026 \ldots$

25. When the space passed over by a moving point is given by $s = ae^t + be^{-t}$, find the velocity and the acceleration, showing that the acceleration is equal to the space.

26. Find the slope of the curve $y = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$ at $x = 0$.

Ans. 1.

27. Find the slope of the curve $y = \dfrac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}$ at $x = 0$. *Ans.* $\frac{1}{a}$.

28. Find the derivative of the implicit function y in $e^{x+y} = xy$. Passing to logarithms:

$$x + y = \log x + \log y. \qquad \frac{dy}{dx} = \frac{y(1 - x)}{x(y - 1)}.$$

29. $x^y = y^x$. Passing to logarithms:

$$y \log x = x \log y. \qquad \frac{dy}{dx} = \frac{y^2 - xy \log y}{x^2 - xy \log x}.$$

30. Find the slope of the probability curve $y = e^{-x^2}$.

Ans. $-2xe^{-x^2}$.

What is the value of the slope at $x = 0$?

Ans. 0.

At $x = 1$?

Ans. $-\frac{2}{e}$.

41. Relative Error. — Since when $y = f(x)$, the relative rate of increase of y is

$$\frac{\frac{dy}{dx}}{y} = \frac{\frac{df(x)}{dx}}{f(x)} = \frac{f'(x)}{f(x)},$$

where $\qquad dy = f'(x)\,dx;$

hence, $\qquad \Delta y = f'(x)\,\Delta x, \qquad (1)$

and $\qquad \dfrac{\Delta y}{y} = \dfrac{f'(x)}{f(x)}\Delta x, \qquad (2)$

are approximate relations. The relation (1) is useful in finding the error in the result of a computation due to a small error in the observed data upon which the computation is based. The relation (2) gives approximately the *relative error* $\frac{\Delta y}{y}$.

1. Thus, to find an expression for the relative error in the volume of a sphere calculated from a measurement of the diameter when there is an error in the measurement. Here $V = \frac{1}{6}\pi D^3$;

$$\therefore \quad \frac{\Delta V}{V} = \left(\frac{1}{2}\pi D^2\right)\frac{\Delta D}{\frac{1}{6}\pi D^3} = 3\frac{\Delta D}{D}.$$

Hence, an error of one per cent in the measurement of the diameter gives approximately an error of three per cent in the calculated volume.

2. Again, from the formula for kinetic energy $K = \frac{1}{2}mv^2$, to show that a small change in v involves approximately twice as great a relative change in K. Here

$$\frac{\Delta K}{K} = m\frac{v\,\Delta v}{\frac{1}{2}mv^2} = 2\frac{\Delta v}{v}.$$

3. If a square is laid out 100 ft. on a side and the tape is 0.01 ft. too long, an error of $\frac{1}{100}$ of one per cent, the relative error in the area is, approximately,

$$\frac{\Delta A}{A} = 2x\frac{\Delta x}{x^2} = 200\frac{(.01)}{(100)^2} = .0002,$$

or $\frac{2}{100}$ of one per cent.

42. The Compound Interest Law. — The $\underset{n=\infty}{\text{limit}}\left(1+\frac{x}{n}\right)^n = e^x$, in Art. 36, arises in a variety of problems. When a function has the general form

$$y = ae^{bx}, \tag{1}$$

then
$$\frac{dy}{dx} = bae^{bx} = by;$$

that is, the rate of change of the function is proportional to the function itself. Many of the changes that occur in nature are in accordance with this law, called by Lord Kelvin *the compound interest law.* It is so called because of the fact that the amount of a sum of money at compound interest has a rate of change at any value proportional to that value, when the change is continuous.

Let A = amount, r = rate per cent, P = principal, and t number of years; then,

$$A = P(1 + r)^t, \text{ when interest is compounded yearly;}$$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}, \text{ at } n \text{ equal intervals each year;}$$

$$A = \lim_{n=\infty} P\left[\left(1 + \frac{r}{n}\right)^n\right]^t = Pe^{rt} \text{ (by Art. 36)} \tag{2}$$

is the amount when the interest is compounded continuously. Here $A = Pe^{rt}$ and $\frac{dA}{dt} = rPe^{rt} = rA$, hence, the rate of change of the A is proportional to the value of A, the factor of proportionality being the rate per cent at which the interest is reckoned. As a comparison, it may be noted that \$1.00 will amount to \$2.594, in ten years with interest at 10 per cent, compounded yearly; while the amount will be \$2.718 when compounded continuously.

If in $A = Pe^{rt}$, t increase in any arithmetical progression, whose common difference is h, A will increase in a geometrical progression whose common ratio is e^{rh}; for if t become $t + h$, A will become $Pe^{r(t+h)}$, that is, Ae^{rh}. Hence A is a quantity which is equally multiplied in equal times.

The density of the air towards the sea level from an elevation is a quantity which is equally multiplied in equal distances of descent, for the increase in density per foot of descent is due to the weight of that layer of air which is itself proportional to the density. Many other instances occur in physics.

When bacteria grow freely the increase per second in the number in a cubic inch of culture is proportional to the number present. The relation between the number N and the time t is expressed by the equation,

$$N = Ce^{kt}; \quad \therefore \quad \frac{dN}{dt} = kCe^{kt} = kN, \tag{3}$$

where N is the number of thousand per cubic inch, and k is the rate of increase shown by a colony of one thousand per cubic inch. So many instances of this kind are found in organic growth — where the rate of growth grows as the total grows — that the law is called *the law of organic growth*, as well as *the compound interest law.*

When a quantity has a rate of change which is proportional to the quantity itself, if the functional relation is expressed by an equation, it must be of the form (1).

In the case of the density of the air, the relation (see Art. 226)* between the density ρ and the height h above the sea level is expressed by

$$\rho = \rho_0 e^{-kh}, \quad \therefore \quad \frac{d\rho}{dh} = -k\rho_0 e^{-kh} = -k\rho, \tag{4}$$

where ρ_0 is the density at the sea level and k is a constant to be determined by experiment. From barometric observations at different altitudes, it has been found that at the height of $3\frac{1}{2}$ miles above the earth's surface, the air is about one-half as dense as it is at the surface. Hence, to determine k,

$$\frac{\rho}{\rho_0} = e^{-3.5k} = \frac{1}{2};$$

$$\therefore \quad -3.5\,k = \log 0.5, \quad \text{or} \quad k = \frac{\log 0.5}{-3.5} = 0.198;$$

$$\therefore \rho = \rho_0 e^{-0.198h}, \text{ where } h \text{ is in miles.} \tag{5}$$

Here, as h increases in arithmetical progression, ρ decreases in geometrical progression, the force of gravity and the temperature being taken constant. The varying density at different heights is found by giving values to h; thus, making $h = 35$, gives $\frac{\rho}{\rho_0} = 0.001$; hence, according to this law at the height of 35 miles the density of the air is about one-thousandth of the density at the sea level. As the pressure p is

* Applied Calculus.

proportional to the density, $p = k'\rho$; and the same law holds for the pressure of the air; hence,

$$p = p_0 e^{-k'h}, \quad \therefore \quad \frac{dp}{dh} = -kp_0 e^{-k'h} = -k'p, \tag{6}$$

where k' is a constant to be found by experiment.

Knowing the pressure at the sea level and observing the pressure at some height, k' is determined; or it can be determined from the value of the pressure at any two differing heights. When the pressure is expressed in inches of mercury in a barometer, the pressure in lbs. per square inch = 0.4908 × barometer reading in inches. Taking $p_0 = 30''$ when $h = 0$, and $p = 24''$, say, when $h = 5830$ ft., k' is readily computed. In millimeters the equation is $p = 760\, e^{-h/8000}$, where h is in meters.

The relation between the decomposition of radium and time is expressed by the equation

$$q = q_0 e^{-kt}; \quad \therefore \quad \frac{dq}{dt} = -kq_0 e^{-kt} = -kq, \tag{7}$$

where q_0 is the original quantity and q is the quantity remaining after a time t. The constant k can be found from the fact that half the original quantity disappears in 1800 years.

The relation between the varying difference of temperature of a body and that of the surrounding medium and the time of cooling is expressed, according to Newton's Law, by

$$\delta = \delta_0 e^{-kt}; \quad \therefore \quad \frac{d\delta}{dt} = -k\delta_0 e^{-kt} = -k\delta, \tag{8}$$

where $\delta = \tau - \tau_0$, the difference in the temperature of the body and that of the medium, $\delta_0 = \tau_1 - \tau_0$, the difference when $t = 0$, k a constant; that is, $\tau = \tau_0 + (\tau_1 - \tau_0)\, e^{-kt}$, where $-kt$ indicates the body is cooling.

TRIGONOMETRIC FUNCTIONS.

43. Circular or Radian Measure. — The formulas for differentiation of trigonometric functions are simpler when the angle is measured in radians than in degrees. Hence, in the formulas that follow, the angle will be in radians.

A *radian*, the unit of circular measure, is an angle which when placed at the center of a circle intercepts an arc equal in length to the radius.

Since $2\pi r$ is arc of 360°, a radian equals $\frac{180°}{\pi}$, or 57.3° nearly. In circular or radian measure, an angle in radians is equal to the length of the intercepted arc divided by the radius; $\theta = \frac{s}{r}$, where θ is angle in radians, s is number of units in arc, and r is the number of units in radius. Hence, $s = r\theta$; that is, in any circle the length of an arc equals the product of the measure of its subtended central angle in radians and the length of the radius. If $r = 1$, then $s = \theta$; that is, the arc and the angle have the same numerical measure. Trigonometric functions are called *circular* functions.

44. Formulas and Rules for Differentiation. —

[XI] $$d^* \sin\theta = \cos\theta\, d\theta.$$

The differential of the sine of an angle is the cosine of the angle by the differential of the angle.

[XII] $$d \cos\theta = -\sin\theta\, d\theta.$$

The differential of the cosine of an angle is minus the sine of the angle by the differential of the angle.

[XIII] $$d \tan\theta = \sec^2\theta\, d\theta.$$

The differential of the tangent of an angle is the secant squared of the angle by the differential of the angle.

[XIV] $$d \cot\theta = -\operatorname{cosec}^2\theta\, d\theta.$$

* Parenthesis after d omitted when no ambiguity results.

The differential of the cotangent of an angle is minus the cosecant squared of the angle by the differential of the angle.

[XV] $$d \sec \theta = \sec \theta \tan \theta \, d\theta.$$

The differential of the secant of an angle is the secant of the angle by the tangent of the angle by the differential of the angle.

[XVI] $$d \operatorname{cosec} \theta = - \operatorname{cosec} \theta \cot \theta \, d\theta.$$

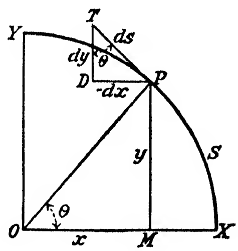
The differential of the cosecant of an angle is minus the cosecant of the angle by the cotangent of the angle by the differential of the angle.

45. Derivation of [XI] and [XII]. —

I. Let the point $P(x, y)$ move along the arc XPY of a unit circle. Denote the number of linear units in the arc XP by s, and the number of *radians* in angle XOP by θ.

Then $$\theta = s, \qquad y = \sin\theta, \qquad x = \cos\theta;$$

$$\therefore \quad d\theta = ds, \quad dy = d\sin\theta, \quad dx = d\cos\theta.$$



Angle DTP equals θ, and dx is negative; hence, from the triangle DTP, by (d) Art. 10,

$dy = d\sin\theta = \cos\theta \, d\theta$, since $ds = d\theta$;

$dx = d\cos\theta = -\sin\theta \, d\theta$,

since $-dx = \sin\theta \, ds$ and $ds = d\theta$.

It is seen that dy and dx in the figure are what the changes of the sine and cosine of θ would be if, at the value XOP of θ, the changes were to become uniform.

II. By the general method of limits.

Let $\quad y = \sin\theta, \quad$ then $\quad y + \Delta y = \sin(\theta + \Delta\theta)$;

$$\Delta y = \sin(\theta + \Delta\theta) - \sin\theta = 2\sin\frac{\Delta\theta}{2}\cos\left(\theta + \frac{\Delta\theta}{2}\right), \text{ by Trig.};$$

$$\frac{\Delta y}{\Delta\theta} = \frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\cos\left(\theta + \frac{\Delta\theta}{2}\right). \quad \text{As} \quad \left(\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right) \doteq 1, \text{ as } \Delta\theta \doteq 0; \quad \text{[(Art. 46).}$$

$$\therefore \quad \frac{dy}{d\theta} = \lim_{\Delta\theta \doteq 0}\left[\frac{\Delta y}{\Delta\theta}\right] = \lim_{\Delta\theta \doteq 0}\left(\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right)\cos\left(\theta + \frac{\Delta\theta}{2}\right) = \cos\theta;$$

$$\therefore \quad dy = d\sin\theta = \cos\theta\, d\theta.$$

Corollary. — d covers $\theta = d\,(1 - \sin\theta) = -\cos\theta\, d\theta$.

Now let $\quad x = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right);$

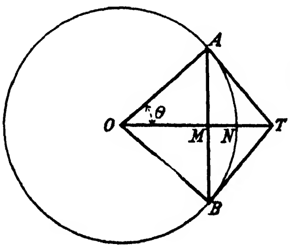
$$dx = d\cos\theta = d\sin\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \theta\right) d\left(\frac{\pi}{2} - \theta\right);$$

$$\therefore \quad d\cos\theta = -\sin\theta\, d\theta.$$

Corollary.— d vers $\theta = d\,(1 - \cos\theta) = \sin\theta\, d\theta$.

46. $$\lim_{\theta \doteq 0}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

Let θ be the number of radians in the angle NOA, where the angle is taken acute; by Geometry, if AT and BT are tangents at A and B,



then,

$$\text{chord } AB < \text{arc } AB < AT + BT,$$

and therefore

$$MA < \text{arc } NA < AT.$$

Hence

$$\frac{MA}{OA} < \frac{\text{arc } NA}{OA} < \frac{AT}{OA};$$

that is,

$$\sin\theta < \theta < \tan\theta;$$

dividing by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \quad \text{or} \quad 1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Thus the ratio $\frac{\sin\theta}{\theta}$ lies between 1 and $\cos\theta$.

When θ approaches 0 as its limit, $\cos\theta$ approaches 1 as its limit; therefore, also $\sin\theta/\theta$ approaches 1 as its limit. (See Art. 154, Ex. 1.)

Corollary. — Since $\lim\limits_{\theta \doteq 0} \dfrac{\sin\theta}{\theta} = 1,$

$$\therefore \quad \lim_{\theta \doteq 0} \frac{2\,MA}{2\,NA} = \lim_{\theta \doteq 0} \frac{\text{chord}\,AB}{\text{arc}\,AB} = 1. \quad \text{(Art. 22, also.)}$$

It may be noted that,

since $\dfrac{MA}{AT} = \dfrac{\sin\theta}{\tan\theta} = \cos\theta,$ when $\theta \doteq 0$ and $\cos\theta \doteq 1,$ $\sin\theta$ and $\tan\theta$ approach equality; and, since the arc θ is intermediate in value between $\sin\theta$ and $\tan\theta$, the three functions approach equality as the angle θ nears zero. So

$$\lim_{\theta \doteq 0}\left(\frac{\sin\theta}{\tan\theta}\right) = 1 \text{ and } \lim_{\theta \doteq 0}\left(\frac{\tan\theta}{\theta}\right) = 1, \text{ as well as } \lim_{\theta \doteq 0}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

These are fundamental examples of the ratio of infinitesimals approaching a constant value as a limit. Consider again the equality of ratios, $\dfrac{MA}{AT} = \dfrac{OM}{OA}\cdot$ Suppose the points A and B approach N; so long as A and B are not coincident, that is, so long as AB is really a chord, the equality still exists. The ratio $MA : AT$ may be considered a function of OM, or equally well a function of the angle NOA. As OM approaches ON as its limit, or as the angle NOA approaches zero as a limit, the ratio $MA : AT$ approaches 1 as its limit. The nearer OM gets to ON, or the nearer A gets to N, the nearer does the ratio $MA : AT$ get to unity. The crucial fact is that the reasoning is vitiated if OM becomes actually equal to ON; for then the triangles will cease to exist, the terms of the one ratio will be zero and those of the other will be identical, and the equation on which the reasoning is based could not be established.

47. Derivation of [XIII]. —

Since $\qquad \tan\theta = \dfrac{\sin\theta}{\cos\theta}, \quad d\tan\theta = d\left(\dfrac{\sin\theta}{\cos\theta}\right);$

$$\therefore \quad d\tan\theta = \frac{\cos\theta\, d\sin\theta - \sin\theta\, d\cos\theta}{\cos^2\theta}$$

$$= \frac{(\cos^2\theta + \sin^2\theta)\, d\theta}{\cos^2\theta} = \sec^2\theta\, d\theta.$$

48. Derivation of [XIV]. —

Since $\qquad \cot\theta = \tan\left(\dfrac{\pi}{2} - \theta\right);$

$$\therefore \quad d\cot\theta = d\tan\left(\frac{\pi}{2} - \theta\right) = \sec^2\left(\frac{\pi}{2} - \theta\right) d\left(\frac{\pi}{2} - \theta\right)$$

$$= -\operatorname{cosec}^2\theta\, d\theta.$$

49. Derivation of [XV]. —

Since $\qquad \sec\theta = \dfrac{1}{\cos\theta};$

$$\therefore \quad d\sec\theta = d\left(\frac{1}{\cos\theta}\right) = + \frac{\sin\theta\, d\theta}{\cos^2\theta}$$

$$= + \sec\theta\tan\theta\, d\theta.$$

50. Derivation of [XVI]. —

Since $\qquad \operatorname{cosec}\theta = \sec\left(\dfrac{\pi}{2} - \theta\right);$

$$\therefore \quad d\operatorname{cosec}\theta = d\sec\left(\frac{\pi}{2} - \theta\right) = \sec\left(\frac{\pi}{2} - \theta\right)\tan\left(\frac{\pi}{2} - \theta\right) d\left(\frac{\pi}{2} - \theta\right)$$

$$= -\operatorname{cosec}\theta\cot\theta\, d\theta.$$

Note. — In the derivations of the formulas for the cosine, cotangent, and cosecant, as given, it may be noted that, as in the last example of Art. 31, the formula for the derivative of the function of a function has appropriate application. Thus for $\cos\theta$, let $x = \cos\theta = \sin\left(\dfrac{\pi}{2} - \theta\right)$ and $\phi = \dfrac{\pi}{2} - \theta,$

then, $$\frac{dx}{d\theta} = \frac{dx}{d\phi} \cdot \frac{d\phi}{d\theta},$$

or $$\frac{d}{d\theta}\cos\theta = \frac{d}{d\phi}\sin\phi \times \frac{d}{d\theta}\left(\frac{\pi}{2} - \theta\right) = -\cos\phi = -\cos\left(\frac{\pi}{2} - \theta\right);$$

$$\therefore \quad \frac{d}{d\theta}\cos\theta = -\sin\theta \quad \text{or} \quad d\cos\theta = -\sin\theta\, d\theta.$$

In practice the actual substitution of the auxiliary symbol ϕ may be dispensed with.

The formula applies to such functions as $y = \sin(ax + b)$. Thus put $z = ax + b$, making $y = \sin z$; then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

or $$\frac{d}{dx}\sin(ax + b) = \frac{d}{dz}\sin z \times \frac{d}{dx}(ax + b) = \cos z \cdot a;$$

$$\therefore \quad \frac{d}{dx}\sin(ax + b) = a\cos(ax + b)$$

or $$d\sin(ax + b) = a\cos(ax + b)\, dx.$$

Again, let the function be $y = \sin^2(ax + b)$ and put $z = \sin(ax + b)$; then

$$\begin{aligned} \frac{d}{dx}\sin^2(ax + b) &= \frac{d}{dz}(z^2) \cdot \frac{d}{dx}\sin(ax + b) \\ &= 2z \times a\cos(ax + b) \\ &= 2a\sin(ax + b)\cos(ax + b); \end{aligned}$$

$$\therefore \quad d\sin^2(ax + b) = 2a\sin(ax + b)\cos(ax + b)\, dx.$$

51. Note on [XI]. — If the angle is measured in degrees, then $d\sin\theta = \frac{\pi}{180}\cos\theta\, d\theta$, since θ degrees is $\frac{\pi\theta}{180}$ radians

and $$\sin\theta^\circ = \sin\left(\frac{\pi\theta^r}{180}\right);$$

$$\begin{aligned} \therefore \quad d\sin\theta^\circ &= d\sin\left(\frac{\pi\theta^r}{180}\right) \\ &= \frac{\pi}{180}\cos\left(\frac{\pi\theta^r}{180}\right)d\theta = \frac{\pi}{180}\cos\theta^\circ\, d\theta. \end{aligned}$$

It is thus seen that the formulas for differentiation of the trigonometric functions are simpler when the angle is measured in radians than when measured in degrees. For the same reason that Napierian or natural logarithms are employed in differentiation, radian or circular measure is used for angles of the trigonometric functions, when differentiation is to be done.

52. Remarks on [XI]. — The fundamental limit of Art. 46, $\lim\limits_{\theta \doteq 0}\left(\frac{\sin\theta}{\theta}\right) = 1$ means that when θ is a small number $\sin\theta$ is approximately equal to θ. For angle of 1°, $\theta = \frac{\pi}{180} = 0.0174533\ldots$, $\sin\theta = 0.0174524\ldots$; so they are equal to five decimals. Of course for angle of 1′ or 1″, they are equal to a great many more decimals, but they are never exactly equal however small the angle may be, since the sine is always less than the arc.

Since $d\sin\theta = \cos\theta\,d\theta$, if the value 0° is taken for θ and $1°\left(=\frac{\pi}{180}\right)$ for $d\theta\,(=\Delta\theta)$,

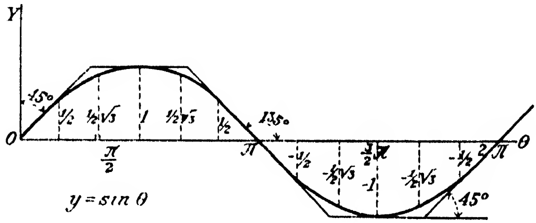
$$d\sin\theta = \cos 0°\frac{\pi}{180} = .0174533\ldots \quad \text{or} \quad \frac{d\sin\theta}{d\theta} = 1,$$

$$\Delta\sin\theta = \sin(0° + \Delta\theta) - \sin 0° = .0174524\ldots = \sin 1°;$$

$$\therefore\quad \frac{\Delta\sin\theta}{\Delta\theta} = \frac{.0174524\ldots}{.0174533\ldots} < 1, \quad \left.\lim_{\Delta\theta \doteq 0}\frac{\Delta\sin\theta}{\Delta\theta} = \frac{d\sin\theta}{d\theta} = 1\right]_{\theta=0°}.$$

From $\frac{d\sin\theta}{d\theta} = \cos\theta = \cos 0° = 1$, it is seen that, at $\theta = 0°$, the sine of θ is changing at the same rate as θ is changing; so the slope of the curve $y = \sin\theta$ is unity at the origin, and the tangent to the curve at that point makes an angle of 45° with θ-axis. The conditions are the same at $\theta = 2\pi$. As $\frac{d\sin\theta}{d\theta} = \cos\theta = \cos 90° = 0$, at $\theta = \frac{\pi}{2}$, the sine of θ is not changing,

the rate being zero, and the tangent to the curve at that point is parallel to θ-axis. At $\theta = \pi$, $\cos 180^\circ = -1$, so the sine of θ is decreasing, at that value of θ, at the same rate as θ is increasing, and the tangent to the curve at that point makes angle of 135° with θ-axis. Thus the rate of change of the sine of θ, at any value of θ, can be found; and the differen-



tial of the sine, the change if the change became uniform, will always differ from the increment, the actual change of the sine, when the angle is given an increment. In taking sines or other functions from tables by interpolation, the changes are assumed as uniform within allowable limits of error.

EXERCISE V.

1. $y = \sin x^2$. $\quad dy = 2x \cos x^2\, dx$.
2. $y = \sin^2 x$. $\quad dy = 2 \sin x \cos x\, dx = \sin 2x\, dx$.
3. $y = \cos ax$. $\quad dy = -a \sin ax\, dx$.
4. $y = f(\theta) = \tan^m \theta$. $\quad \dfrac{dy}{d\theta} = f'(\theta) = m \tan^{m-1} \theta \sec^2 \theta$.
5. $f(\theta) = \tan 3\theta + \sec 3\theta$. $\quad f'(\theta) = 3 \sec^2 3\theta + 3 \sec 3\theta \tan 3\theta$.
6. $f(x) = \sin(\log ax)$. $\quad f'(x) = 1/x \cos(\log ax)$.
7. $f(x) = \log(\sin ax)$. $\quad f'(x) = a \cot ax$.
8. $y = \dfrac{\sin x + \cos x}{e^x}$. $\quad \dfrac{dy}{dx} = -\dfrac{2 \sin x}{e^x}$.
9. $f(\theta) = \log(\tan a\theta)$. $\quad f'(\theta) = \dfrac{2a}{2 \sin(a\theta)\cos(a\theta)} = \dfrac{2a}{\sin(2a\theta)}$.
10. $f(\theta) = \log(\cot o\theta)$. $\quad f'(\theta) = -2a/\sin(2a\theta)$.
11. $f(\theta) = \tan(\log \theta)$. $\quad f'(\theta) = 1/\theta \sec^2(\log \theta)$.
12. $f(\theta) = \log(\sec \theta)$. $\quad f'(\theta) = \dfrac{\sec \theta \tan \theta}{\sec \theta} = \tan \theta$.

13. $f(x) = x^{\sin x}$. $\qquad f'(x) = x^{\sin x}(\sin x/x + \log x \cdot \cos x)$.

14. $f(x) = (\sin x)^x$. $\qquad f'(x) = (\sin x)^x (x \cot x + \log \sin x)$.

15. $f(\theta) = (\sin\theta)^{\tan\theta}$. $\qquad f'(\theta) = (\sin\theta)^{\tan\theta}(1 + \sec^2\theta \log\sin\theta)$.

16. $f(\theta) = \frac{1}{3}\tan^3\theta - \tan\theta + \theta$. $\qquad f'(\theta) = \tan^4\theta$.

17. $f(x) = \tan\sqrt{1-x}$. $\qquad f'(x) = \dfrac{-(\sec\sqrt{1-x})^2}{2\sqrt{1-x}}$.

18. $f(\theta) = \log\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$. $\qquad f'(\theta) = \csc\theta$.

By differentiation derive each of the following pairs of identities from the other:

19. $\sin 2\theta \equiv 2\sin\theta\cos\theta$, $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$.

20. $\sin 2\theta \equiv \dfrac{2\tan\theta}{1+\tan^2\theta}$, $\quad \cos 2\theta \equiv \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$.

21. $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$,
$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

22. $\sin(m+n)\theta \equiv \sin m\theta\cos n\theta + \cos m\theta\sin n\theta$,
$\cos(m+n)\theta \equiv \cos m\theta\cos n\theta - \sin m\theta\sin n\theta$

23. If θ vary uniformly, so that 360° is described in π seconds, show that the rates of increase of $\sin\theta$, when $\theta = 0°, 30°, 45°, 60°, 90°$, are respectively, 2, $\sqrt{3}$, $\sqrt{2}$, 1, 0, per second. (See figure, Art 52.)

53. The Sine Curve or Wave Curve. — The locus of the equation

$$y = \sin x, \tag{1}$$

where x is an angle in radians, is called the *sine curve*, from its equation, or the *wave curve*, from its shape. The maximum value of y is called the *amplitude*, being unity in (1); and, since the curve is unchanged when $x + 2\pi$ is substituted for x, the curve $y = \sin x$ is a periodic curve with a *period* equal to 2π. (See figure, Art. 52, and figure, Art. 73.)

The more general form of the equation is

$$y = a\sin mx, \tag{2}$$

where a is the amplitude and $\dfrac{2\pi}{m}$ is the period, m a constant.

The curve is called the *sinusoid* also, and is of great importance, since it is the *type form* of the fundamental waves of science; such as, sound waves, vibrations of rods,

wires, plates and bridge members, tidal waves in the ocean, and ripples on a water surface. The ordinary progressive waves of the sea are *not* of this shape, as they have the form of a *trochoid.*

54. Damped Vibrations. — When a body vibrates in a medium like a gas or liquid, the amplitude of the swings get smaller and smaller, or the motion slowly (or rapidly in some cases) dies out. Thus, when a pendulum vibrates in the air the rate of decay of the amplitude is quite slow; but when in oil the rate is rapid. The ratio between the lengths of the successive amplitudes of vibration is called the *damping factor* or the *modulus of decay.*

In all such cases the amplitude of the swings *differ by a constant amount* or the *logarithmic decrement is constant.* Hence the amplitude must satisfy an equation of the form

$$A = ae^{-bt}, \tag{1}$$

where A is the amplitude and t the time. The actual motion is given by an equation of the form

$$y = ae^{-bt} \sin \omega t, \quad \text{where} \quad \omega = \frac{v}{a} \text{ is a constant.} \tag{2}$$

(See Art. 73.)

* In plotting a curve whose equation is of this form, say,

$$y = e^{-\frac{1}{4}x} \sin \frac{\pi}{2} x, \tag{3}$$

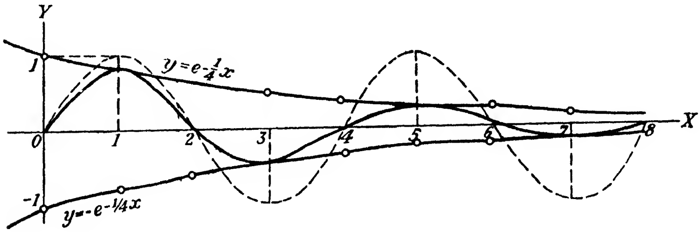
much is gained by the following considerations:

1. Since the numerical value of the sine never exceeds unity the values of y in (3) will not exceed in numerical value the value of the first factor $e^{-\frac{1}{4}x}$. As the extreme values of $\sin \frac{1}{2}\pi x$ are $+1$ and -1, *y has the extreme values $e^{-\frac{1}{4}x}$* and $-e^{-\frac{1}{4}x}$. Hence, if the curves

$$y = e^{-\frac{1}{4}x} \quad \text{and} \quad y = -e^{-\frac{1}{4}x} \tag{4}$$

* This illustration is given substantially in Smith and Gales's *New Analytic Geometry.*

are drawn, the locus of (3) will lie entirely *between* these curves. They are called *boundary curves,* and they are plotted by three or more points, the second being symmetrical to the first with respect to the x-axis.



2. When $\sin \frac{1}{2}\pi x = 0$, then in (3) $y = 0$, since the first factor is always finite. Hence, the locus of (3) meets the x-axis in the same points as the sine curve

$$y = \sin \tfrac{1}{2}\pi x. \tag{5}$$

3. The required curve is tangent to the boundary curves when the second factor, $\sin \frac{1}{2}\pi x$, is $+1$ or -1; that is, when the ordinates of the curve (5) have a maximum or a minimum value. The tangency is proven by finding the derivative of y in (3) and noting that, when $\sin \frac{1}{2}\pi x$ is $+1$ or -1, it will be the same as the derivatives of y in (4). Hence, the slopes of the curves and the ordinates being equal for the same values of x, the required curve is tangent to one or the other of the boundary curves for those values of x that make $\sin \frac{1}{2}\pi x = +1$ or -1. Thus, differentiating (3) and (4) gives

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{4}e^{-\frac{1}{4}x}\sin\frac{1}{2}\pi x + \frac{\pi}{2}e^{-\frac{1}{4}x}\cos\frac{1}{2}\pi x\\ &= -\tfrac{1}{4}e^{-\frac{1}{4}x}, \text{ when } \sin\tfrac{1}{2}\pi x = 1,\\ &= \tfrac{1}{4}e^{-\frac{1}{4}x}, \text{ when } \sin\tfrac{1}{2}\pi x = -1.\end{aligned}$$

For the sine curve (5) the period is 4 and the amplitude is 1. This curve is the broken line of the figure.

The locus of (3) *crosses the x-axis at* $x = 0, \pm 2, \pm 4, \pm 6,$

etc., *and is tangent to the boundary curves* (4) *at* $x = \pm 1, \pm 3, \pm 5$, etc. The discussion having disclosed these facts, the curve is readily sketched, as in the figure; that is, the winding curve between the boundary curves (4).

A more general form of the equation of a damped vibration is

$$y = ae^{-bt} \sin (\omega t - \alpha), \text{ where } \alpha = \frac{\omega t_0}{a} \text{ is constant.} \tag{3'}$$

This equation may be written either (see Art. 73)

$$y = e^{-bt} (A \sin kt + B \cos kt),$$

where A and B are constants,

or
$$y = A \sin (\omega t - \alpha), \text{ where } A = ae^{-bt}. \tag{3''}$$

Here A is a *variable* decreasing amplitude, whose relative rate of decrease is $-dA/dx \div A = b$; that is, the *relative rate of decrease of A is constant.*

The successive derivatives from (3′) are (by Art. 68):

$$\frac{dy}{dt} = ae^{-bt} [- b \sin (\omega t - \alpha) + \omega \cos (\omega t - \alpha)],$$

$$\frac{d^2y}{dt^2} = ae^{-bt} [b^2\omega^2 \sin (\omega t - \alpha) - 2 b\omega \cos (\omega t - \alpha)],$$

whence it follows that

$$\frac{d^2y}{dt^2} + 2 b \frac{dy}{dt} + (b^2 + \omega^2) y = 0. \tag{4'}$$

Equations which contain derivatives or differentials are called *differential equations.* The equation (4′) is the fundamental differential equation for damped vibrations. The term in $\frac{dy}{dt}$, *or v, proportional to the velocity*, occurs in equations for vibration only when damping is considered. Vibrations are cases of simple harmonic motion — damping being caused by resistances, such as friction, etc. Simple harmonic motion is treated in Art. 73.

INVERSE TRIGONOMETRIC FUNCTIONS.

55. Formulas and Rules for Differentiation. — The direct trigonometric functions are single-valued but the angle has to be restricted to a certain range in order that the inverse functions may be single valued. To make the inverse functions single-valued, the angle denoted by $\sin^{-1} x$, $\operatorname{cosec}^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\operatorname{covers}^{-1} x$, is taken to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and the angle denoted by $\cos^{-1} x$, $\sec^{-1} x$, $\operatorname{vers}^{-1} x$, to lie between 0 and π. Thus

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right); \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6};$$

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}; \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2};$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \text{ if } x \text{ be positive,}$$

$$= -\frac{\pi}{2}, \text{ if } x \text{ be negative.}$$

These restrictions will be assumed in the following formulas, and all will be expressed in terms of the letter x. While the symbols $\sin^{-1} x$ and $\operatorname{arc} \sin x$ are both used to denote the *angle whose sine is* x, in writing the formulas the notation $\sin^{-1} x$ is preferable.

[XVII] $$d \sin^{-1} x = \frac{dx}{\sqrt{1 - x^2}}.$$

The differential of an angle in terms of its sine is the differential of the sine divided by the square root of one minus the square of the sine.

[XVIII] $$d \cos^{-1} x = -\frac{dx}{\sqrt{1 - x^2}}.$$

The differential of an angle in terms of its cosine is minus the differential of the angle in terms of its sine.

[XIX] $$d \tan^{-1} x = \frac{dx}{1 + x^2}.$$

The differential of an angle in terms of its tangent is the differential of the tangent divided by one plus the square of the tangent.

[XX] $$d \cot^{-1} x = -\frac{dx}{1+x^2}.$$

The differential of an angle in terms of its cotangent is minus the differential of the angle in terms of its tangent.

[XXI] $$d \sec^{-1} x = \frac{dx}{x\sqrt{x^2-1}}.$$

The differential of an angle in terms of its secant is the differential of the secant divided by the secant and the square root of the square of the secant minus one.

[XXII] $$d \operatorname{cosec}^{-1} x = -\frac{dx}{x\sqrt{x^2-1}}.$$

The differential of an angle in terms of its cosecant is minus the differential of the angle in terms of its secant.

[XXIII] $$d \operatorname{vers}^{-1} x = \frac{dx}{\sqrt{2x-x^2}}.$$

The differential of an angle in terms of its versine is the differential of the versine divided by the square root of twice the versine minus the square of the versine.

[XXIV] $$d \operatorname{covers}^{-1} x = -\frac{dx}{\sqrt{2x-x^2}}.$$

The differential of an angle in terms of its coversine is minus the differential of the angle in terms of its versine.

56. Derivation of [XVII] and [XVIII]. —

Let $\theta = \sin^{-1} x$; then $\sin\theta = x$, the differential of which by [XI] is $\cos\theta \, d\theta = dx$;

$$\therefore \quad d\theta = \frac{dx}{\cos\theta} = \frac{dx}{\sqrt{1-\sin^2\theta}} = \frac{dx}{\sqrt{1-x^2}}.$$

Now $$d\cos^{-1} x = d\left(\frac{\pi}{2} - \sin^{-1} x\right) = -\frac{dx}{\sqrt{1-x^2}}.$$

57. Derivation of [XIX] and [XX]. —

Let $\theta = \tan^{-1} x$; then $\tan\theta = x$, the differential of which by [XIII] is $\sec^2\theta \, d\theta = dx$;

$$\therefore \quad d\theta = \frac{dx}{\sec^2\theta} = \frac{dx}{1+\tan^2\theta} = \frac{dx}{1+x^2}.$$

Now $$d\cot^{-1} x = d\left(\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{dx}{1+x^2}.$$

58. Derivation of [XXI] and [XXII]. —

Let $\theta = \sec^{-1} x$; then $\sec\theta = x$, the differential of which by [XV] is $\sec\theta\tan\theta \, d\theta = dx$;

$$\therefore \quad d\theta = \frac{dx}{\sec\theta\tan\theta} = \frac{dx}{x\sqrt{\sec^2\theta - 1}}$$
$$= \frac{dx}{x\sqrt{x^2-1}}.$$

Now $$d\operatorname{cosec}^{-1} x = d\left(\frac{\pi}{2} - \sec^{-1} x\right) = -\frac{dx}{x\sqrt{x^2-1}}.$$

59. Derivation of [XXIII] and [XXIV]. —

Let $\theta = \operatorname{vers}^{-1} x$; then $\operatorname{vers}\theta = x$, the differential of which by *Cor.*, Art. 45, is $\sin\theta \, d\theta = dx$;

$$\therefore \quad d\theta = \frac{dx}{\sin\theta} = \frac{dx}{\sqrt{1-\cos^2\theta}} = \frac{dx}{\sqrt{1-(1-\operatorname{vers}\theta)^2}}$$
$$= \frac{dx}{\sqrt{1-(1-x)^2}} = \frac{dx}{\sqrt{2x-x^2}}.$$

Now $$d\operatorname{covers}^{-1} x = d\left(\frac{\pi}{2} - \operatorname{vers}^{-1} x\right) = -\frac{dx}{\sqrt{2x-x^2}}.$$

EXERCISE VI.

1. $d\sin^{-1}\dfrac{x}{a} = \dfrac{d(x/a)}{\sqrt{1-(x/a)^2}} = \dfrac{dx}{\sqrt{a^2-x^2}}.$

2. $d\cos^{-1}\dfrac{x}{a} = \dfrac{-dx}{\sqrt{a^2-x^2}}$; $\qquad d\tan^{-1}\dfrac{x}{a} = \dfrac{a\,dx}{a^2+x^2}$;

$d\cot^{-1}\dfrac{x}{a} = \dfrac{-a\,dx}{a^2+x^2}$; $\qquad d\sec^{-1}\dfrac{x}{a} = \dfrac{a\,dx}{x\sqrt{x^2-a^2}}$;

$d\csc^{-1}\dfrac{x}{a} = \dfrac{-a\,dx}{x\sqrt{x^2-a^2}}$; $\qquad d\operatorname{vers}^{-1}\dfrac{x}{a} = \dfrac{dx}{\sqrt{2ax-x^2}}.$

Note. — These may be considered standard formulas.

3. $y = \tan x \tan^{-1} x.$ $\quad \dfrac{dy}{dx} = \sec^2 x \tan^{-1} x + \dfrac{\tan x}{1+x^2}.$

4. $y = \tan^{-1} \dfrac{2x}{1+x^2}.$ $\quad \dfrac{dy}{dx} = \dfrac{2(1-x^2)}{1+6x^2+x^4}.$

5. $y = \sin^{-1} \dfrac{x+1}{\sqrt{2}}.$ $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-2x-x^2}}.$

6. $y = \text{arc} \sin \sqrt{\sin x}.$ $\quad \dfrac{dy}{dx} = \dfrac{\sqrt{1+\csc x}}{2}.$

7. $y = x^{\sin^{-1} x}.$ $\quad \dfrac{dy}{dx} = x^{\sin^{-1} x} \left(\dfrac{\sin^{-1} x}{x} + \dfrac{\log x}{\sqrt{1-x^2}} \right).$

8. $y = \tan^{-1} (n \tan x).$ $\quad \dfrac{dy}{dx} = \dfrac{n}{\cos^2 x + n \sin^2 x}.$

9. $y = \text{arc} \cos \dfrac{e^x - e^{-x}}{e^x + e^{-x}}.$ $\quad \dfrac{dy}{dx} = \dfrac{-2}{e^x + e^{-x}}.$

10. $y = \text{arc vers} \dfrac{2x^2}{1+x^2}.$ $\quad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}.$

11. $y = \text{arc} \tan \dfrac{3x - x^3}{1 - 3x^2}.$ $\quad \dfrac{dy}{dx} = \dfrac{3}{1+x^2}.$

12. $y = \text{arc} \sin \dfrac{1-x^2}{1+x^2}.$ $\quad \dfrac{dy}{dx} = \dfrac{-2}{1+x^2}.$

13. $y = \text{arc} \tan \dfrac{x+a}{1-ax}.$ $\quad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}.$

14. $\phi = \text{arc} \tan \dfrac{dy}{dx}.$ $\quad \dfrac{d\phi}{dx} = \dfrac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}.$

15. $y = \tan^{-1} \dfrac{x}{\sqrt{1-x^2}}.$ $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}.$

16. $y = \text{arc} \sin \dfrac{2}{e^x + e^{-x}}.$ $\quad \dfrac{dy}{dx} = \dfrac{-2}{e^x + e^{-x}}.$

17. $y = \text{arc} \tan (\sec x + \tan x).$ $\quad \dfrac{dy}{dx} = \dfrac{1}{2}.$

18. $y = \text{arc} \sin \left(\dfrac{\sin x - \cos x}{\sqrt{2}} \right).$ $\quad \dfrac{dy}{dx} = 1.$

19. $y = \tan^{-1} \dfrac{3x-2}{5} + \cot^{-1} \dfrac{3x-12}{6x+1}.$ $\quad \dfrac{dy}{dx} = 0.$

20. $y = \text{arc} \cot \dfrac{e^{ax} + e^{-ax}}{e^{ax} - e^{-ax}}.$ $\quad \dfrac{dy}{dx} = \dfrac{2a}{e^{2ax} + e^{-2ax}}.$

21. What is the slope of the curve $y = \sin x$? Its inclination lies between what values? What is its inclination at $x = 0$? What at $x = \pi/2$?

The slope $= \cos x$; hence, at any point, it must have a value between -1 and $+1$, inclusive. Hence, the inclination of the curve at any point is between 0 and $\pi/4$ or between $3\pi/4$ and π, inclusive. (See figure, Art. 52.)

60. Hyperbolic Functions. — These are certain functions, recognized as far back as 1757, that have been introduced in recent years, and that are coming more and more into use.

As the trigonometric functions are called circular because of their relation to the circle, the hyperbolic are so called because of their relation to the rectangular hyperbola, the relations being in some respects the same. The functions are analogous to the trigonometric functions and their names are the same. They are the hyperbolic sine, cosine, tangent, etc., and they are defined as follows:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \operatorname{csch} x = \frac{1}{\sinh x},$$

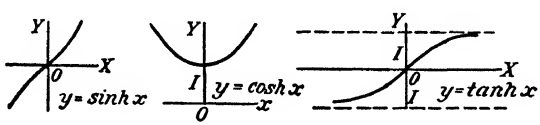
$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \operatorname{sech} x = \frac{1}{\cosh x},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

61. General Relations. — Besides the reciprocal relations given above, the same as those between the circular functions, there are analogous relations:

$$\cosh^2 x - \sinh^2 x = 1; \quad 1 - \tanh^2 x = \operatorname{sech}^2 x;$$
$$\coth^2 x - 1 = \operatorname{csch}^2 x; \quad \sinh 2x = 2 \sinh x \cosh x;$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 - 1 = 1 + 2\sinh^2 x.$$

62. Numerical Values. Graphs. — The sine may have any value from $-\infty$ to ∞; the cosine any value from 1 to



∞; the tangent any value between -1 and 1, and the lines whose equations are $y = \pm 1$ are asymptotes to the graph of $\tanh x$. The graphs of the sine and cosine are both asymptotic to the graph of $y = \frac{1}{2} e^x$.

63. Derivatives. — Since $\frac{d}{dx} e^x = e^x$ and $\frac{d}{dx} e^{-x} = -e^{-x}$, by differentiating the hyperbolic functions as functions of x, the several derivatives are readily found to be as follows:

$$\frac{d}{dx} \sinh x = \cosh x, \qquad \frac{d}{dx} \cosh x = \sinh x;$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x; \qquad \frac{d}{dx} \coth x = -\operatorname{cosech}^2 x;$$

$$\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \coth x;$$

$$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x.$$

The differentials are given at once by the derivatives, or *vice versa;* thus, $d \sinh x = \cosh x \, dx$, and so for the others.

64. The Catenary. — The curve $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$ is called the catenary and is important because it is the curve of a perfectly flexible and inextensible cord between two points, and is the curve that a material cable when hung between two supports is assumed to take.

Since $\frac{d}{dx} \cosh x = \sinh x$, $\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x}) = \sinh x$ is the slope of the catenary. The general equation of the catenary is

$$y = a \cosh \frac{x}{a} = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

where a is the distance from the origin to the lowest point of the curve. (See Art. 146, Applied Calculus.)

65. Inverse Functions. — The inverse functions are useful when expressed as logarithms.

If $y = \sinh^{-1} x$, the logarithmic form of y is found from

$$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y}),$$

which is reduced to

$$e^{2y} - 2\,xe^y - 1 = 0;$$

solving as a quadratic gives

$$e^y = x \pm \sqrt{x^2+1};$$

but as e^y is always positive,

$$e^y = x + \sqrt{x^2+1}; \quad \therefore \quad \sinh^{-1} x = y = \log\left(x + \sqrt{x^2+1}\right).$$

In the same way is found, $\cosh^{-1} x = \log\left(x \pm \sqrt{x^2-1}\right)$.

Since $$\left(x - \sqrt{x^2-1}\right) = \frac{1}{\left(x + \sqrt{x^2-1}\right)};$$

$$\therefore \quad \log\left(x - \sqrt{x^2-1}\right) = -\log\left(x + \sqrt{x^2-1}\right).$$

For each value of x greater than 1 there are two values of $\cosh^{-1} x$, equal numerically but of opposite sign.

In the same way again is found,

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}, \quad \text{if } x^2 < 1;$$

$$\coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1}, \quad \text{if } x^2 > 1.$$

66. Derivatives of Inverse Functions. — The derivatives of the inverse functions are found by differentiating their logarithmic forms, using formula $\dfrac{d}{dx}\log x = \dfrac{1}{x}\cdot$

The derivatives, taking $\dfrac{x}{a}$ instead of x, are:

$$\frac{d}{dx}\sinh^{-1}\frac{x}{a} = \frac{1}{\sqrt{x^2+a^2}};$$

$$\frac{d}{dx}\cosh^{-1}\frac{x}{a} = \pm\frac{1}{\sqrt{x^2-a^2}};$$

$$\frac{d}{dx}\tanh^{-1}\frac{x}{a} = \frac{a}{a^2-x^2}, \quad (x^2 < a^2);$$

$$\frac{d}{dx}\coth^{-1}\frac{x}{a} = \frac{-a}{x^2-a^2}, \quad (x^2 > a^2).$$

Since the inverse cosine is not single-valued, for the positive ordinate of $\cosh^{-1}\frac{x}{a}$, the $+$ sign must be taken. When $\frac{x}{a}$ is used instead of x,

$$\sinh^{-1}\frac{x}{a} = \log\frac{x+\sqrt{x^2+a^2}}{a}$$
$$= \log\left(x+\sqrt{x^2+a^2}\right) - \log a,$$

so the derivative of $\sinh^{-1}\frac{x}{a}$ is the same as that of $\log\left(x+\sqrt{x^2+a^2}\right)$, since $d(\log a) = 0$. The divisor a occurs in the logarithmic form of $\cosh^{-1}\frac{x}{a}$ also, so its presence should be borne in mind when comparing the same result expressed in logarithms and in inverse hyperbolic sines or cosines.

The relation of the inverse hyperbolic sine to the equilateral hyperbola is shown in Art. 137, and the inverse functions are again considered in Art. 120.

Expansion of cosh x/a and sinh x/a. — Expanding $e^{x/a}$ and $e^{x/a}$ in series and taking the sum of the two series term by term gives

$$y = a\cosh\frac{x}{a} = a\left[1+\frac{x^2}{a^2\,2!}+\frac{x^4}{a^4\,4!}+\cdots\right]$$
$$= a+\frac{x^2}{2\,a}+\frac{x^4}{24\,a^3}+\cdots. \qquad (10)$$

Taking the difference of the two series gives

$$s = a\sinh\frac{x}{a} = a\left[\frac{x}{a}+\frac{x^3}{a^3\,3!}+\frac{x^5}{a^5\,5!}+\cdots\right]$$
$$= x+\frac{x^3}{6\,a^2}+\frac{x^5}{120\,a^4}+\cdots; \qquad (11)$$

$$\therefore\quad \tan\phi = \left[\frac{x}{a}+\frac{x^3}{a^3\,3!}+\frac{x^5}{a^5\,5!}+\cdots\right]$$
$$= \frac{x}{a}+\frac{x^3}{6\,a^3}+\frac{x^5}{120\,a^5}+\cdots. \qquad (12)$$

For these expansions, put $\frac{x}{a}$ for x in Examples 8 and 7 of Exercise XXX.

CHAPTER III.

SUCCESSIVE DIFFERENTIATION. ACCELERATION. CURVILINEAR MOTION.

67. Successive Differentials. — It is often desired to differentiate the differential of a variable or to get the derivative of a derivative. For, while the differential of the independent variable, being arbitrary, is usually supposed to have the same value at all values of the variable and hence to be a constant, the differential of the dependent variable, except when the function is linear, is a variable, subject to differentiation.

The differential of dy is called the *second* differential of y; the differential of the second differential of y is called the *third* differential of y; and so on. $d(dy)$ is written d^2y; $d(d^2y)$ or $dd\,dy$, is written d^3y; and so on. The figure written like an exponent to d denotes how many times in succession the operation of differentiation has been performed. dy, d^2y, d^3y, . . . d^ny are called the *successive differentials* of y.

Example. — The successive differentials of y when $y = ax^3$:

$$\begin{aligned} dy &= 3\,ax^2\,dx; \\ d^2y &= 3\,a\,dx \cdot d\,(x^2) = 6\,ax\,dx^2; \\ d^3y &= 6\,a\,dx^2 \cdot dx = 6\,a\,dx^3; \\ d^4y &= d\,(6\,a\,dx^3) = 0. \end{aligned}$$

The independent variable being x, dx is treated as a constant. Note that according to the notation adopted $d^2y \equiv d\,dy$; $dy^2 \equiv (dy)^2$; $d\,(y^2) \equiv 2\,y\,dy$.

68. Successive Derivatives. — The derivative of the first derivative of a function is called the *second derivative* of the function; the derivative of the second derivative is called the *third derivative;* and so on.

When x is independent,

$$\frac{d}{dx}\frac{dy}{dx} \equiv \frac{d^2y}{dx^2}, \quad \frac{d}{dx}\frac{d^2y}{dx^2} \equiv \frac{d^3y}{dx^3}, \quad \ldots, \quad \frac{d}{dx}\frac{d^{n-1}y}{dx^{n-1}} \equiv \frac{d^ny}{dx^n}.$$

The successive derivatives of $f(x)$ are denoted by

$$f'(x), \quad f''(x), \quad f'''(x), \quad f^{\text{IV}}(x), \ldots, f^n(x).$$

Thus if $\quad f(x) = x^4, \quad f'(x) = 4x^3, \quad f''(x) = 12x^2,$

$$f'''(x) = 24x, \quad f^{\text{IV}}(x) = 24, \quad f^{\text{V}}(x) = 0.$$

Hence, if $y = f(x)$ and x is independent,

$$\frac{dy}{dx} = f'(x), \quad \frac{d^2y}{dx^2} = f''(x), \ldots, \frac{d^ny}{dx^n} = f^n(x).$$

The *nth* derivative of some functions can be easily found by inspection of a few of the derivatives.

Example 1.—$f(x) = e^x, f'(x) = e^x, f''(x) = e^x, f'''(x) = e^x,$

$$\ldots, \quad \therefore \quad f^n(x) = e^x.$$

This function e^x is remarkable in that its rate of change, or derivative, is equal to the function itself.

Example 2.—$f(\theta) = \sin\theta$, $f'(\theta) = \cos\theta$, $f''(\theta) = -\sin\theta$, $f'''(\theta) = -\cos\theta$, $f^{\text{IV}}(\theta) = \sin\theta$. $f'(\theta) = \cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$, $f''(\theta) = \cos\left(\theta + \frac{\pi}{2}\right) = \sin\left(\theta + 2\cdot\frac{\pi}{2}\right)$, $f'''(\theta) = \cos(\theta + \pi)$ $= \sin\left(\theta + 3\cdot\frac{\pi}{2}\right) \ldots; \quad \therefore f^n(\theta) = \sin\left(\theta + n\cdot\frac{\pi}{2}\right)$.

Each of the successive derivatives of $f(x)$ equals the x-rate of the preceding derivative, for $f^n(x) = \frac{d}{dx}f^{n-1}(x)$ = the x-rate of $f^{n-1}(x)$.

Corollary. — $f^{n-1}(x)$ is an increasing or a decreasing function of x according as $f^n(x)$ is positive or negative, and conversely.

Note. — The tangential acceleration is,

$$a_t = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2},$$

and the flexion is

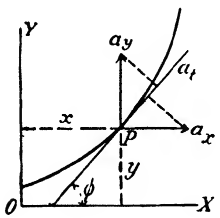
$$b = \frac{dm}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}.$$

(See Art. 12 and Art. 13.) Hence, the speed $\frac{ds}{dt}$ is increasing or decreasing according as the acceleration $\frac{d^2s}{dt^2}$ is positive or negative, and the slope is increasing or decreasing according as the flexion $\frac{d^2y}{dx^2}$ is plus or minus.

When the second derivatives are equal to zero, the first derivatives are constant, or conversely. (See Art. 13.)

69. Resolution of Acceleration. — An acceleration, like a velocity, being a quantity which has magnitude and direction, may be represented by a straight line, that is, by a *vector*.

In general the acceleration a at any point (x, y) of a curvilinear path may be resolved into two components in given directions. The directions usually taken are along the tangent and normal at the point, and in directions parallel to rectangular axes OX, OY. With the notation of the figure for (d), Art. 10, the components parallel to the axes being the rates of change of dx/dt and dy/dt will be denoted by d^2x/dt^2 and d^2y/dt^2, respectively. The rate of change of the *velocity* is the resultant acceleration



$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}.$$

To find the component acceleration a_t along the tangent at P; resolve the axial accelerations along the tangent, giving for the sum of tangential components,

$$a_t = \frac{d^2x}{dt^2}\cos\phi + \frac{d^2y}{dt^2}\sin\phi = \frac{d^2x}{dt^2}\cdot\frac{dx}{ds} + \frac{d^2y}{dt^2}\cdot\frac{dy}{ds} = \frac{d^2s}{dt^2},$$

by differentiating,

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2;$$

$$\therefore \quad a_t = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \text{rate of change of the } \textit{speed}.$$

Hence, the tangential component is the same as for rectilinear motion.

EXERCISE VII.

Find dy, d^2y, d^3y, when:

1. $y = 2x^5 - 5x^4 + 20x^3 - 5x^2 + 2x.$
 $d^3y = 120(x^2 - x + 1)\,dx^3.$

2. $y = x^2 \log(x - 1).$ $\qquad d^3y = \dfrac{2(x^2 - 3x + 3)}{(x-1)^3}\,dx^3.$

3. $y = (x^2 - 6x + 12)\,e^x.$ $\qquad d^3y = x^2e^x\,dx^3.$

4. $y = \log \sin x.$ $\qquad d^3y = 2\cos x \sin^{-3} x\,dx^3.$

5. $y = \tan x.$ $\qquad d^3y = (6\sec^4 x - 4\sec^2 x)\,dx^3.$

Find the successive derivatives:

6. $f(x) = x^6 + 4x^4 + 3x + 2.$ $\qquad f^{\text{VI}}(x) = \underline{|6}, \quad f^{\text{VII}}(x) = 0.$

7. $f(x) = \log(1 + x)$; find nth derivative.

$$f'(x) = (1 + x)^{-1}, \quad f''(x) = (-1)(1 + x)^{-2},$$

$$f'''(x) = (-1)^2\underline{|2}\,(1 + x)^{-3}, \quad f^{\text{IV}}(x) = (-1)^3\underline{|3}\,(1 + x)^{-4}, \ldots$$

$$\therefore \quad f^n(x) = (-1)^{n-1}\,\underline{|n-1}\,(1 + x)^{-n}.$$

8. $f(x) = x^3 \log x.$ $\qquad f^{\text{IV}}(x) = 6x^{-1}.$

9. $y = \log(e^x + e^{-x}).$ $\qquad \dfrac{d^3y}{dx^3} = -8\dfrac{e^x - e^{-x}}{(e^x + e^{-x})^3}.$

10. Find formula, known as Leibnitz's theorem, for $d^n(uv)$.

Let u and v be functions of x; then

$$d(uv) = du \cdot v + u\,dv, \tag{1}$$

$$d^2(uv) = d^2u \cdot v + du\,dv + du\,dv + u\,d^2v$$

$$= d^2u \cdot v + 2\,du\,dv + ud^2v; \tag{2}$$

$$\therefore \quad d^3(uv) = d^3u \cdot v + 3\,d^2u\,dv + 3\,du\,d^2v + u\,d^3v. \tag{3}$$

The coefficients and *exponents* of *differentiation* are according to the Binomial theorem, however far the differentiation is continued;

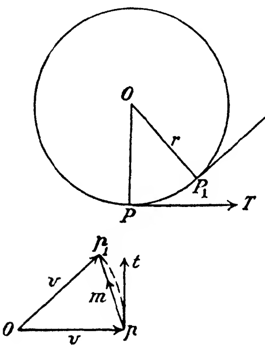
$$\therefore \quad d^n(uv) = d^nu \cdot v + nd^{n-1}u\,dv + \frac{n(n-1)}{\underline{|2}}\,d^{n-2}u\,d^2v + \cdots$$

$$+ udu\,d^{n-1}v + u\,d^n v.$$

11. If $x^2 + y^2 = a^2$, $\dfrac{d^2y}{dx^2} = -\dfrac{a^2}{y^3}$. **12.** If $y^2 = 2\,px$, $\dfrac{d^2y}{dx^2} = -\dfrac{p^2}{y^3}$.

13. If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$. **14.** If $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$.

70. Circular Motion. — When a point describes a circle of radius r with constant speed v, it has a constant acceleration v^2/r directed towards the center of the circle.



Let PT be the velocity at P, and P_1T_1 that at P_1. A velocity being a directed quantity may be represented by a *vector;* that is, by a straight line whose length denotes magnitude and whose direction is the given direction. Hence from a common origin o, the vectors op and op_1 are drawn equal to the vectors PT and P_1T_1, respectively. Since the speed is constant each vector is v, and pp_1 is the vector increment, denoted by Δv. The average acceleration for the interval of time Δt is $\dfrac{\Delta v}{\Delta t}$ directed along pp_1, and is laid off as pm.

As Δt approaches zero, P_1 approaches P, and p_1 approaches p along the circular arc indicated by the dotted line; pm approaches a vector pt directed along the tangent to the arc pp_1 at p. This vector, the $\lim\limits_{\Delta t \doteq 0}\left[\dfrac{\Delta v}{\Delta t}\right]$, represents the acceleration $\dfrac{dv}{dt}$ of the point P moving in the circle of radius r; and since the direction is at right angles with the tangent at P, the acceleration is directed towards the center O, is normal acceleration, therefore, denoted by a_n. To find the magnitude of the normal acceleration a_n: since the sectors pop_1 and POP_1 are similar, the angles at o and O being equal,

$$\frac{\text{arc } pp_1}{op} = \frac{\text{arc } PP_1}{OP}, \quad \text{or} \quad \frac{\text{arc } pp_1}{v} = \frac{\Delta s}{r};$$

$$\therefore \quad \frac{\text{arc } pp_1}{\Delta t} = \frac{v}{r}\frac{\Delta s}{\Delta t}, \quad \text{and} \quad \lim_{\Delta t \doteq 0}\left[\frac{\text{arc } pp_1}{\Delta t}\right] = \frac{v}{r}\lim_{\Delta t \doteq 0}\left[\frac{\Delta s}{\Delta t}\right];$$

replacing the arc pp_1 by its chord, (Art. 22.)

$$\lim_{\Delta t \doteq 0}\left[\frac{\text{chord } pp_1}{\Delta t}\right] = \frac{dv}{dt} = \frac{v}{r}\frac{ds}{dt},$$

$$\therefore \quad a_n = \frac{dv}{dt} = \frac{v^2}{r}. \tag{1}$$

Otherwise, it may be seen that while the point P describes the circle of radius r, the point p describes the circle of radius v, the velocity of p in its path being the acceleration of P in its path. Since the circles are described in the same time, the velocities are to each other as the paths of the two points, or as the radii of the circles.

$$\therefore \quad \frac{\text{velocity of } p}{v} = \frac{\text{velocity of } P}{r};$$

velocity of p is $\frac{dv}{dt}$, rate of change of velocity v of P,

$$\therefore \quad \frac{a_n}{v} = \frac{v}{r}, \quad a_n = \frac{v^2}{r}.$$

Since the *speed* is constant, the rate of change a_t is zero,

$\therefore \quad a = \sqrt{a_t^2 + a_n^2} = a_n = \frac{v^2}{r}$; that is, the total acceleration is the normal acceleration, the change of *velocity* being change of direction *only*. Since

$$s = 2\pi r = vt; \quad a = \frac{4\pi^2 r}{T^2}, \tag{2}$$

where T is time of a revolution.

Note. — By Newton's Second Law the measure of the force on a moving body is $\frac{W}{g}a$ (Art. 71); hence, the force acting on a body weighing W lbs. revolving in a circle of

radius r is $\dfrac{Wv^2}{gr}$ pounds of force, is directed towards the center of the circle, and is called *centripetal force.* The reaction of the body to this force is by the Third Law equal in magnitude and opposite in direction. It acts upon the *axis* or upon *whatever* deflects the body from its otherwise rectilinear path, and has been called the *centrifugal force,* although a misnomer. The centripetal force is the active force, the other is the equal and opposite *reaction* and should be called the *centrifugal reaction,* since it is the resistance which the inertia of the body opposes to the force acting upon it.

71. The Second Law of Motion. -- According to Newton's Second Law of Motion the rate of change of momentum of a body is proportional to the resultant of the impressed forces acting on the body.

Let a body of standard weight W be moving with velocity v, then Wv = momentum of the body;

$$\therefore \quad \frac{d}{dt}(Wv) = W\frac{dv}{dt} = W\frac{d^2s}{dt^2} = Wa_t.$$

Hence, if F be the resultant force and a the acceleration, by the Law,

$$Wa \propto F \quad \text{or} \quad Wa = kF; \tag{1}$$

that is, the product of the numbers representing the weight and the acceleration is proportional to the number representing the force. The value of the factor k depends upon the units used for the other factors. When these are the usual units, foot, pound (weight), second, pound (force), it is found by experiment that k has the value 32, approximately. Experiment shows that, while the value varies slightly for different localities, it is the same for all bodies in any one locality. This value is denoted by g and is called the acceleration of gravity; for when a body falls freely, gravity being the only force acting, the acceleration is found to be about

32 feet per sec. per sec. The locality in which $g = g_0 = 32.1740$ ft./sec.^{2} has been adopted as the "standard locality" and the weight of the body in that locality is called the *standard weight* of the body. Putting g for k in equation (1) it becomes

$$Wa = gF \quad \text{or} \quad F = \frac{W}{g}a. \tag{2}$$

If the force F is the force of gravity acting on W, then

$$\frac{dv}{dt} = a = g,$$

the acceleration of gravity; for the weight W is the force of gravity acting on the body denoted by the letter W.

Since for any given body the ratio of the force to the acceleration produced is constant, the value of this ratio, F/a or W/g, is a characteristic of the body, called its *inertia* and the ratio may be denoted by the letter m; then the equation (2) may be written

$$F = ma. \tag{3}$$

In using equation (3) for the solution of problems, with the usual units, m must be replaced by W/g.

Some writers use the word "mass" to denote the inertia, while others use it for standard weight; consequently, there are some who avoid the use of the word on account of the resulting confusion.

In Physics the equation (3) is used, the unit of force, called the absolute unit, being that unit which in equation (1) makes $k = 1$, the other units remaining the same, and, therefore, m measured in pounds the same numerically as the standard weight W.

Accordingly, if the number g be the number of absolute units of force with which gravity attracts the unit mass (or weight), the Law becomes

$$m\frac{dv}{dt} = mg, \quad \text{hence} \quad \frac{dv}{dt} = g, \text{ the } \textit{acceleration of gravity.}$$

The absolute unit of force is thus, *that force, which acting on the unit of mass (or weight) for the unit of time, generates the unit of velocity.* The absolute unit of force is thus $1/g$ of a pound avoirdupois, about $\frac{1}{2}$ of an ounce, and F is given in this unit when in

$$F = ma,$$

m is expressed in pounds, the unit being a pound. The ordinary unit of force, sometimes called the Engineer's unit, is *one pound* and is g times the absolute unit used in Physics.

Newton's Second Law of Motion gives as a definition of force: *force is the time-rate of change of momentum.* Using the much abused term "mass," the definition is: *the force is the product of the mass times the acceleration.* From

$$F = \frac{W}{g} a; \tag{2}$$

$$F_t = \frac{W}{g}\frac{d^2s}{dt^2}, \qquad F_n = \frac{W}{g}\frac{v^2}{r},$$

$$F_x = \frac{W}{g}\frac{d^2x}{dt^2}, \qquad F_y = \frac{W}{g}\frac{d^2y}{dt^2},$$

the tangential, normal, and axial components of a force F, corresponding to the accelerations, a_t, a_n, a_x, a_y. Since kinetic energy of a moving body is $E = \frac{1}{2} mv^2$,

$$\therefore \quad \frac{dE}{dv} = mv,$$

that is, the v-rate of E is momentum;

$$\frac{dE}{dt} = mv\frac{dv}{dt} = Fv,$$

that is, the time-rate of E is product of force and velocity.

72. Angular Velocity and Acceleration. — When a body is rotating about an axis the amount of rotation depends upon the time; so if θ is the angle through which any line in the body, intersecting the axis at right angles, turns, then θ

gives the amount of rotation and is a function of the time t. Thus in the case of a wheel the rotation is measured by the angle θ through which a spoke turns in a time t. The rotation is uniform if the body rotates through equal angles in equal intervals of time. The rate of rotation or the rate of change of the angle is the angular velocity or speed and is denoted by ω.

If the rotation is uniform, the angular velocity is constant and $\omega = \frac{\theta}{t}$, θ being in radians; hence, if the uniform rate of rotation is ω radians per second, the body rotates through ωt radians in t seconds of time.

If the rotation is not uniform the rate at which the body is rotating at any instant is the angular velocity at that time,

and
$$\omega = \lim_{\Delta t = 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}.$$

This expression for angular velocity is general and is applicable when the rotation is uniform also; for then,

$$\omega = \frac{\theta}{t} = \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt},$$

although, the ratios being constant, no limit is involved.

Similarly, the angular acceleration is $\alpha = \frac{\omega}{t}$, for constant acceleration; and

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

is, in general, the time-rate of change of angular velocity, or the angular acceleration.

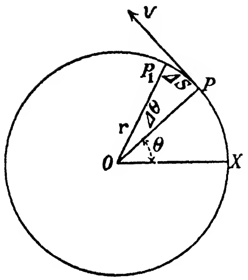
If a particle is at a distance r from the axis of rotation, the relation between the angular velocity of the particle and its linear velocity follows at once, whether the rotation is uniform or not.

Since in circle $\quad \Delta s = r \cdot A\theta,$

$$\frac{\Delta\theta}{\Delta t} = \frac{1}{r} \cdot \frac{\Delta s}{\Delta t},$$

$$\lim_{\Delta t \doteq 0} \frac{\Delta\theta}{\Delta t} = \frac{1}{r} \cdot \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t}.$$

$$\therefore \quad \omega = \frac{d\theta}{dt} = \frac{1}{r} \cdot \frac{ds}{dt} = \frac{v}{r},$$



the relation sought. Hence, since the angular velocity of every point of a rotating body has the same value at any instant, and the direction of motion of a particle at any point is along the tangent at the point,

$$\text{tangential velocity } v = r\omega,$$

if ω is the angular velocity of the particle about axis at 0.

Since $\quad \dfrac{dv}{dt} = \dfrac{d\,(r\omega)}{dt}, \quad \text{or} \quad \dfrac{d^2s}{dt^2} = r\dfrac{d^2\theta}{dt^2},$

$$\therefore \quad \text{tangential acceleration } a_t = r\alpha,$$

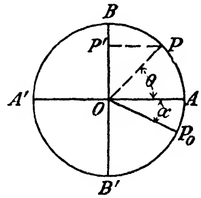
the relation between tangential acceleration and angular acceleration when α is the angular acceleration of a particle at a distance r from the axis of rotation.

Since $\quad a_n = \dfrac{dv}{dt} = \dfrac{v^2}{r} = \dfrac{r^2\omega^2}{r} = r\omega^2,$

$$\therefore \quad \text{normal acceleration } a_n = r\omega^2.$$

73. Simple Harmonic Motion. — If a point move uniformly on a circle and the point be projected on any straight line in the plane of the circle, the back-and-forth motion of the projected point on the given straight line is called *simple harmonic motion.* It is denoted by the letters S. H. M. Let the point P move upon the circumference of a circle of radius a with the uniform velocity of v feet per second, so

that the radius OP rotates with uniform angular velocity at the rate of $\frac{v}{a} = \omega$ radians per second. The projection, P', of P on the vertical diameter, moves up and down. Let θ be the angle that the radius makes with the x-axis, then if the point P was at A when $t = 0$, the displacement $OP' = y$ is given by



$$y = a \sin \theta = a \sin \omega t.$$

If the point P was at P_0 when $t = 0$, and at A when $t = t_0$,

then $$y = a \sin (\omega t - \alpha), \quad \text{where} \quad \alpha = \omega t_0 = \frac{vt_0}{a}. \tag{1}$$

When the displacement at time t is given by (1) the motion is S. H. M. Hence, the point P' describes S. H. M.

The velocity of a point describing S. H. M. is, from (1),

$$\frac{dy}{dt} = a\omega \cos (\omega t - \alpha), \tag{2}$$

and the acceleration is

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin (\omega t - \alpha) \tag{3}$$

$$= -\omega^2 y, \text{ from (1)}, \tag{4}$$

or $$\frac{d^2y}{dt^2} + \omega^2 y = 0. \tag{5}$$

It should be noted that equation (1) may be written in the form

$$y = a \sin (\omega t - \alpha) = a [\sin \omega t \cos \alpha - \cos \omega t \sin \alpha]$$

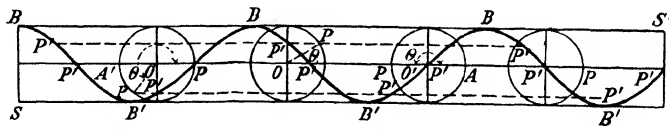
or $$y = A \sin kt + B \cos kt,$$

where $A = a \cos \alpha$ and $B = -a \sin \alpha$ are constants. This equation and $y = a \sin (kt - \alpha)$ are the general formulas for S. H. M.

The acceleration of a particle describing S. H. M., as shown by (4), is proportional to the displacement and

oppositely directed. It is oppositely directed since the motion is one of oscillation about a position of equilibrium. When the body is above this position the force is directed downward, and when it is below, the force is upward. In the figure the point P' has a negative acceleration when above 0 and a positive acceleration when below 0. The acceleration is zero at O, a maximum at B' and a minimum at B; while the corresponding velocity, as given by (2), has its maximum numerical value as P' passes through O in either direction, and is zero at B and B', the ends of the vertical diameter. The factor of proportionality ω^2 is connected with the *period* T by the relation $T = \frac{2\pi}{\omega}$, where the period of the S. H. M. $y = a \sin \omega t$ is the time T required for a complete revolution of the point P; that is, $\omega T = 2\pi$. The time $t_0 = \frac{\alpha}{\omega}$ to make part of a revolution is called the *phase*, α being *epoch angle*. The number of complete periods per unit of time is $N = \frac{1}{T} = \frac{\omega}{2\pi}$, where N is the *frequency* of the S. H. M.

Let P' be a tracing point capable of describing a curve on a uniformly translated sheet of paper, SS', then if the sheet be



moved with the same speed as the point P moves on the circumference of the circle of radius a, P' describing S. H. M. on the vertical diameter will trace the sinusoid $P'BP'B'$ on the moving paper. The sinusoid will have as its equation

$$y = a \sin \frac{v}{a} t = a \sin \frac{x}{a},$$

where x is the abscissa of any point of the sinusoid referred

to an origin (as $0'$) moving with the paper. The circle is shown in the figure in several positions corresponding to the different angles through which the radius OP has revolved, or the different positions of the projected point P' on the vertical diameter BOB'. The amplitude of the S. H. M. is the same as that of the sinusoid; that is, the radius a of the circle. The period of the sinusoid is $2\pi a$, corresponding to the period, $T = \frac{2\pi}{\omega}$, of the S. H. M. of the point P' on the vertical diameter.

74. Self-registering Tide Gauge. — The principle by which the up-and-down motion of a point is represented by a curve is utilized in the self-registering tide gauge for recording the rise and fall of the tide. Such a gauge consists essentially of a float protected by a surrounding house or tube, and attached by suitable mechanism to a pencil that has a motion proportional to the vertical rise and fall of the float. The pencil bears against a piece of graduated paper fastened to a drum that is revolved by clockwork. There will thus be drawn on the paper a curve where the horizontal units are time, and the vertical units are feet of rise and fall. The stage of the tide is given for any time.

EXERCISE VIII.

1. The angle (in radians) through which a rotating body turns, starting from rest, is given by the equation

$$\theta = \tfrac{1}{2}\alpha t^2 + \omega_0 t + \theta_0,$$

where α, ω_0, θ_0 are constants; find the formulas for angular velocity and angular acceleration after any time t.

$$\omega = \frac{d\theta}{dt} = \alpha t + \omega_0, \text{ which gives the angular velocity;}$$

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \alpha, \text{ which gives the angular acceleration.}$$

2. A flywheel is brought from rest up to a speed of 60 revolutions per minute in $\frac{1}{2}$ minute. Find the average angular acceleration α, and

the number of revolutions required. Find the velocity at the end of 15 seconds.

$$\omega = 60 \text{ r.p.m.} = 60 \times \frac{2\pi}{60} = 2\pi \text{ radians per sec.}$$

$$\therefore \quad \alpha t = \alpha \cdot 30 = 2\pi \quad \text{or} \quad \alpha = \frac{2\pi}{30} = 0.2094 \text{ rad./sec}^2.$$

$$\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}\frac{2\pi}{30}(30)^2 = 15 \times 2\pi = 15 \text{ revolutions.}$$

$$\omega = \alpha t = \frac{2\pi}{30} \times 15 = \pi = 3.14 \text{ rad. per sec.}$$

3. If the flywheel of Ex. 2 is 12 feet in diameter, find the tangential velocity and acceleration of a point on the rim. Find the normal acceleration at the instant full speed is attained.

$$v = r\omega = 6 \times 2\pi = 37.7 \text{ ft. per sec.}$$

$$\alpha_t = r\alpha = 6 \times \frac{2\pi}{30} = 6 \times 0.2094 = 1.256 \text{ ft./sec}^2.$$

Illustrative Example.

Assuming that the moon's orbit is circular, its acceleration towards the earth is (by Art. 70 (2)),

$$a = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2} = 0.0089 \text{ ft./sec.}^2,$$

where $R = 238{,}800$ miles and $T = 27.32$ days.

From the law of inverse squares:

$$\frac{a}{g} = \frac{r^2}{R^2} = \frac{r^2}{(60.267\,r)^2} = \frac{1}{3632};$$

$$\therefore \quad a = \frac{g}{3632} = \frac{32.089}{3632} = 0.0088 \text{ ft./sec}^2.,$$

where 32.089 is the value of g on the earth at the equator, and R is 60.267 times r, the radius of the earth.

As these results differ by only $\frac{1}{10000}$th of a foot, the conclusion is that the centripetal force on the moon in its orbit is due to the earth's attraction, acting according to the law of inverse squares. (See Art. 198.)

CHAPTER IV.

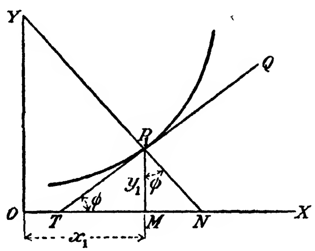
GEOMETRICAL AND MECHANICAL APPLICATIONS.

75. (*a*) **Tangents and Normals.** — Since the derivative $\frac{dy}{dx} = f'(x)$ represents the slope of the curve $y = f(x)$ at any point (x, y),

$$\left[\frac{dy}{dx}\right]_1 = \text{slope of } P_1T = \tan\phi = m_1,$$

where ϕ is the angle XTQ, measured from the positive direction of the x-axis to the tangent TP_1, and m_1 is the slope of the curve at the point $P_1(x_1, y_1)$.

Hence from the equation of a line through a given point (x_1, y_1), $y - y_1 = m(x - x_1)$; the equation of the tangent at the point $P_1(x_1, y_1)$ is $y - y_1 = \left[\frac{dy}{dx}\right]_1 (x - x_1)$, in which the subscript denotes that the quantity is taken with the value which it has at the point P_1. Since the sign of the derivative of a function indicates whether the function is increasing or decreasing, when m_1 is positive the curve is rising at P_1, and when m_1 is negative the curve is falling there. If m_1 is zero the tangent is horizontal, parallel to, or coincident with the x-axis; and if m_1 is infinite, the tangent is vertical, parallel to, or coincident with y-axis.



Points where the slope has any desired value can be found

by setting the derivative equal to the given number and solving the resulting equation for x.

The slope of the normal NP_1, being the negative reciprocal of the slope of the tangent TP_1, is

$$n_1 = -\frac{1}{m_1} = -\cot\phi = \left[-\frac{dx}{dy}\right]_1.$$

Hence, the equation of the normal is

$$y - y_1 = \left[-\frac{dx}{dy}\right]_1 (x - x_1) \quad \text{or} \quad x - x_1 + (y - y_1)\left[\frac{dy}{dx}\right]_1 = 0.$$

(*b*) **Subtangents and Subnormals.** — The subtangent and the subnormal are the projections on the x-axis of the part of the tangent and normal, respectively, between the point of tangency and the x-axis.

From the figure:

$$\text{Subtangent } TM = y_1 \cot\phi = y_1 \left[\frac{dx}{dy}\right]_1.$$

$$\text{Subnormal } MN = y_1 \tan\phi = y_1 \left[\frac{dy}{dx}\right]_1.$$

$$\text{Tangent} \quad TP_1 = \sqrt{\overline{MP_1}^2 + \overline{TM}^2} = \sqrt{y_1^2 + y_1^2\left[\frac{dx}{dy}\right]_1^2}$$

$$= y_1\sqrt{1 + \left[\frac{dx}{dy}\right]_1^2} = y_1 \operatorname{cosec}\phi.$$

$$\text{Normal} \quad NP_1 = \sqrt{\overline{MP_1}^2 + \overline{MN}^2} = \sqrt{y_1^2 + y_1^2\left[\frac{dy}{dx}\right]_1^2}$$

$$= y_1\sqrt{1 + \left[\frac{dy}{dx}\right]_1^2} = y_1 \sec\phi.$$

If the subtangent is reckoned from the point T, and the subnormal from the point M, each will be positive or negative according as it extends to the right or to the left. For any given curve the signs will depend upon the coördinates of the point of tangency.

Note. — As mentioned before, the problem of tangents directly led to the Differential Calculus.

76. Illustrative Examples. — 1. The circle $x^2 + y^2 = a^2$.

Differentiating, $\quad 2\,x\,dx + 2\,y\,dy = 0,$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y},$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_1 = -\frac{x_1}{y_1}.$$

Equation of tangent,

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

or $xx_1 + yy_1 = a^2$, after reducing.

Equation of normal,

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

or $yx_1 - y_1x = 0$, after reducing.

The final form of the last equation shows that the normal at any point on the circle passes through the center.

The subtangent

$$TM = y_1\left[\frac{dx}{dy}\right]_1 = y_1\left(-\frac{y_1}{x_1}\right) = -\frac{y_1^2}{x_1} = -\frac{a^2 - x_1^2}{x_1}.$$

The subnormal $\quad MN = y_1\left[\dfrac{dy}{dx}\right]_1 = y_1\left(-\dfrac{x_1}{y_1}\right) = -x_1.$

Since $dy = -\dfrac{x}{y}\,dx$, the ordinate of the circle changes $-\dfrac{x}{y}$ times as fast as the abscissa; and since $dy = -\dfrac{x}{y}\,dx$ is negative, unless x and y have different signs, y is a decreasing function of x in the first and third quadrants; while dy being positive when the moving point is generating the second and fourth quadrants, y is an increasing function of x in those quadrants.

2. The parabola $y^2 = 2\,px$.

Differentiating, $2\,y\,dy = 2\,p\,dx$,

$$\therefore \quad \frac{dy}{dx} = \frac{p}{y}, \qquad \therefore \quad \left[\frac{dy}{dx}\right]_1 = \frac{p}{y_1}.$$

Equation of tangent, $y - y_1 = \frac{p}{y_1}(x - x_1)$ or $yy_1 = p(x + x_1)$, after reducing.

Equation of normal, $y - y_1 = -\frac{y_1}{p}(x - x_1)$.

The subtangent $TM = y_1\left(\frac{y_1}{p}\right) = \frac{y_1^2}{p} = \frac{2px_1}{p} = 2x_1$.

The subnormal $MT = y_1\left(\frac{p}{y_1}\right) = p$.

Hence, for any point on the parabola the subtangent is bisected at the vertex and the subnormal is constantly equal to p, the semi-latus rectum. These two characteristics of the parabola afford ready methods of accurately drawing a tangent at any point on the curve.

Since $dy = \frac{p}{y}dx$, the rate of $y = \frac{p}{y}$ times rate of x. To find where the rates are the same, put $\frac{dy}{dx} = \frac{p}{y} = 1$, $\therefore\ y = p$ and $x = \frac{p}{2}$; that is, the extremity of the latus rectum is the point where the rates of y and x are equal. Hence the tangents at the extremities of the latus rectum make angles of 45° and 135° with x-axis and meet at right angles with each other at intersection of directrix and x-axis. It is evident that at the origin where $y = 0$, $\frac{dy}{dx} = \infty$; that is, the y-axis is tangent at the vertex. It is seen also that, as y increases without limit, the tangent at its extremity becomes more and more nearly parallel to the x-axis.

3. On the circle $x^2 + y^2 = 1$, to find the points where the slope is 1, 0, or ∞.

$$\left[\frac{dy}{dx}\right]_1 = -\frac{x_1}{y_1} = 1, \quad \therefore \quad y_1 = -x_1;$$

substituting in $x^2 + y^2 = 1$, $x_1 = \pm\frac{1}{2}\sqrt{2}$ and $y_1 = \mp\frac{1}{2}\sqrt{2}$.

$$\left[\frac{dy}{dx}\right]_1 = -\frac{x_1}{y_1} = 0, \quad \therefore \quad x_1 = 0 \quad \text{and} \quad y_1 = \pm 1,$$

by substituting in $x^2 + y^2 = 1$.

$$\left[\frac{dy}{dx}\right]_1 = -\frac{x_1}{y_1} = \infty, \quad \therefore \quad y_1 = 0 \quad \text{and} \quad x_1 = \pm 1,$$

by substituting in $x^2 + y^2 = 1$.

4. To find at what angle the circle $x^2 + y^2 = 8$ and the parabola $y^2 = 2x$ intersect.

Making the two equations simultaneous, the points of intersection are found to be (2, 2) and (2, −2).

For circle, $\quad m_1 = \left[\frac{dy}{dx}\right]_1 = -\frac{x_1}{y_1} = -\frac{2}{\pm 2} = \mp 1.$

For parabola, $m_1 = \left[\frac{dy}{dx}\right]_1 = \frac{p}{y_1} = \frac{1}{\pm 2}.$

For angle of intersection,

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{2}{\pm 2} - \frac{1}{\pm 2}}{1 - \frac{1}{2}} = \mp 3,$$

or $\phi = \tan^{-1}(\mp 3)$, and from table of tangents, $\phi = 108° \, 26'$ or $71° \, 34'$.

5. The path of a point is the arc of a parabola $y^2 = 2px$, and its velocity is v; find its velocity parallel to each axis.

Let s denote the length of the path measured from any point on it; then $\frac{ds}{dt} = v$.

From $y^2 = 2px$,

$$\frac{dy}{dt} = \frac{p}{y}\frac{dx}{dt}.$$

Substituting these values in

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \quad \text{(Art. 11)},$$

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \frac{p^2}{y^2}\left(\frac{dx}{dt}\right)^2,$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{v^2}{1 + \dfrac{p^2}{y^2}} = \frac{y^2v^2}{y^2 + p^2}, \quad \text{or} \quad \frac{dx}{dt} = \frac{yv}{\sqrt{y^2 + p^2}};$$

$$\therefore \quad \frac{dy}{dt} = \frac{p}{y}\frac{dx}{dt} = \frac{pv}{\sqrt{y^2 + p^2}}.$$

6. A comet's orbit is a parabola, and its velocity is v; find its rate of approach to the sun, which is at the focus of its orbit.

Let ρ denote the distance from the focus to any point on $y^2 = 2\,px$; then $\rho = x + \frac{1}{2}\,p$, from point to directrix;

$$\therefore \quad \frac{d\rho}{dt} = \frac{dx}{dt},$$

p being constant. Hence, the comet approaches or recedes from the sun just as fast as it moves parallel to the axis of its orbit;

$$\therefore \quad \frac{d\rho}{dt} = \frac{dx}{dt} = \frac{y}{\sqrt{y^2 + p^2}}\,v. \quad \text{(Example 5.)}$$

At the vertex, $y = 0$; hence, at the vertex $\dfrac{d\rho}{dt}$ is zero. When $y = p$,

$$\frac{d\rho}{dt} = \frac{1}{2}\sqrt{2}\,v.$$

$$\frac{d\rho}{dt} = \frac{dx}{dt} < v, \quad \text{since} \quad \frac{y}{\sqrt{y^2 + p^2}} < 1.$$

$$\frac{dy}{dt} = \frac{pv}{\sqrt{y^2 + p^2}}, \quad \text{(Example 5)}$$

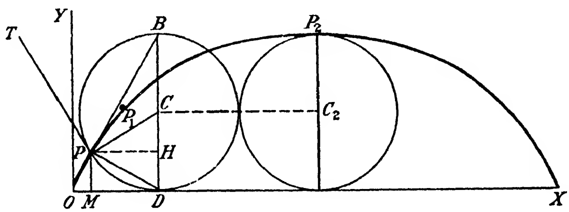
and
$$\lim_{y=\infty} \frac{pv}{\sqrt{y^2 + p^2}} = 0;$$

$$\therefore \quad \frac{dx}{dt} \doteq \frac{ds}{dt} = v, \quad \text{since} \quad \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Hence $\dfrac{d\rho}{dt} = \dfrac{dx}{dt}$ is always less than v and approaches v as a limit as y increases without limit.

7. To compare the velocity of a train moving along a horizontal tangent with the velocity of a point on the flange of one of the wheels, and to compare also the horizontal and vertical components of the flange point.

Let a wheel whose radius is a roll along a horizontal line with a velocity v; find the velocity of any point P on its rim, also the velocity of P horizontally and vertically.



The path of P is a cycloid whose equations are:

$$\left.\begin{aligned} x &= a(\theta - \sin\theta), \\ y &= a(1 - \cos\theta) = a \operatorname{vers}\theta, \end{aligned}\right\} \tag{1}$$

where θ denotes the variable angle DCP, and a the radius CD.

Since the center of the wheel is vertically over D,

$$\begin{aligned} v &= \text{the time-rate of } OD \\ &= \frac{d(a\theta)}{dt} = a \cdot \frac{d\theta}{dt}, \end{aligned}$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{v}{a}. \tag{2}$$

Differentiating equations (1) gives, by (2),

$$\begin{aligned} \frac{dx}{dt} = a(1 - \cos\theta)\frac{d\theta}{dt} &= a \operatorname{vers}\theta \frac{d\theta}{dt} = v \operatorname{vers}\theta \\ &= \text{the velocity horizontally,} \end{aligned} \tag{3}$$

and

$$\frac{dy}{dt} = a \sin\theta \frac{d\theta}{dt} = v \sin\theta = \text{the velocity vertically.} \tag{4}$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = v\sqrt{2(1-\cos\theta)} = v\sqrt{\frac{2y}{a}}$$

$$= \text{velocity of } P \text{ along its path.} \qquad (5)$$

The velocity of P may be considered as the resultant of two velocities each $= v$, one along PT tangent to the circle and the other along PH parallel to the path of C. The resultant PB must bisect the angle HPT; $\therefore$ $DPB = 90°$ and PB is tangent to the cycloid, the path of P, making PD the normal.

At O, $\theta = 0$, and $\frac{dx}{dt} = \frac{dy}{dt} = \frac{ds}{dt} = 0.$

At P, $\theta = \frac{\pi}{3}$, $\frac{dx}{dt} = \frac{1}{2}v$, $\frac{dy}{dt} = \frac{1}{2}v\sqrt{3}$, $\frac{ds}{dt} = v.$

At P_1, $\theta = \frac{\pi}{2}$, $\frac{dx}{dt} = \frac{dy}{dt} = v$, $\frac{ds}{dt} = v\sqrt{2}.$

At P_2, $\theta = \pi$, $\frac{dx}{dt} = \frac{ds}{dt} = 2v$, $\frac{dy}{dt} = 0.$

From (5) is obtained

$$\frac{ds}{dt} : v = \sqrt{2a \cdot y} : a.$$

Hence, the velocity of P is to that of C as the chord DP is to the radius DC; that is, P and C are momentarily moving about D with equal angular velocities. (See Art. 72.)

When $\theta = 60°$, their linear velocities also are equal, as shown above.

8. Find the equation of the tangent and the values of the subnormal and normal of the cycloid.

Dividing (4) by (3), Example 7, gives

$$\frac{dy}{dx} = \frac{\sin\theta}{\operatorname{vers}\theta} = \frac{\sqrt{(2a-y)y/a}}{y/a} = \sqrt{\frac{(2a-y)}{y}},$$

since $\sin\theta = \frac{PH}{CP} = \frac{\sqrt{HB \cdot DH}}{a}$ and $\operatorname{vers}\theta = \frac{y}{a}$ from (1);

$$\therefore \quad \left[\frac{dy}{dx}\right]_1 = \sqrt{\frac{(2a-y_1)}{y_1}}, \text{ and } y - y_1 = \sqrt{\frac{(2a-y_1)}{y_1}}(x-x_1)$$

is the equation of the tangent at point (x_1, y_1).

The subnormal $= y\dfrac{dy}{dx} = y\dfrac{\sin\theta}{\text{vers}\,\theta} = y\dfrac{\sin\theta}{y/a} = a\sin\theta = PH = MD$.

Thus the normal at P passes through the foot of the perpendicular to OX from C. Hence, to draw a tangent and normal at P, locate C, draw the perpendicular DCB equal to $2a$, and join P with B and D; then PB and PD will be respectively the tangent and normal at P.

$$\text{Normal} = DP = \sqrt{DB \cdot DH} = \sqrt{2a \cdot y}.$$

9. Eliminating θ in equations (1) of Example 7, equation of cycloid is

$$x = a \cdot \text{arc vers}\, y/a \mp \sqrt{2ay - y^2},$$

since $\theta = \text{arc vers}\, y/a$ and $a \cdot \sin\theta = \pm\sqrt{(2a-y)a}$.

EXERCISE IX.

Deduce the following equations of the tangent and the normal:

1. The ellipse, $x^2/a^2 + y^2/b^2 = 1$, $x_1x/a^2 + y_1y/b^2 = 1$,

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

2. The hyperbola, $x^2/a^2 - y^2/b^2 = 1$, $x_1x/a^2 - y_1y/b^2 = 1$,

$$y - y_1 = \frac{-a^2y_1}{b^2x_1}(x - x_1).$$

3. The hyperbola, $2xy = a^2$, $x_1y + y_1x = a^2$, $y_1y - x_1x = y_1^2 - x_1^2$.

4. The circle, $x^2 + y^2 = 2ax$, $y - y_1 = (x - x_1)(a - x_1)/y_1$,
$y - y_1 = (x - x_1)\,y_1/(x_1 - r)$.

5. Find the equations of the tangent and normal at $(3/2a, 3/2a)$: $x^3 + y^3 = 3axy$. *Ans.* $x + y = 3a$, $x = y$.

6. $x + y = 2e^{x-y}$, at $(1, 1)$.
Ans. $3y = x + 2$, $3x + y = 4$.

7. $(x/a)^n + (y/b)^n = 2$, at (a, b).
Ans. $x/a + y/b = 2$, $ax - by = a^2 - b^2$.

8. Show that the sum of the intercepts of the tangent to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, is equal to a.

9. Show that the area of the triangle intercepted from the coordinate axes by the tangent to the hyperbola, $2\,xy = a^2$, is equal to a^2.

10. Show that the part of the tangent to the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, intercepted between the axes, is equal to a.

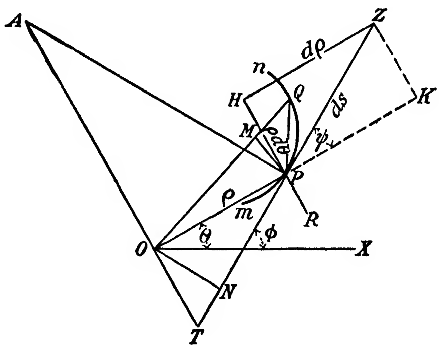
11. Find the slope of the logarithmic curve $x = \log_b y$. The slope varies as what? What is the slope of the curve $x = \log y$?

12. Find the normal, subnormal, tangent, and subtangent of the catenary $y = a/2\,(e^{x/a} + e^{-x/a})$.

$$\textit{Ans.}\ \ y^2/a;\ \ a/4\,(e^{2x/a} - e^{-2x/a});\ \ \frac{y^2}{\sqrt{y^2 - a^2}};\ \ \frac{ay}{\sqrt{y^2 - a^2}}.$$

13. At what angles does the line $3\,y - 2\,x - 8 = 0$ cut the parabola $= 8\,x$? *Ans.* arc tan 0.2; arc tan 0.125.

77. Polar Subtangent, Subnormal, Tangent, Normal. — Let arc $mP = s$, and arc $PQ = \Delta s$; then $\angle\ POQ = \Delta\theta$,



circular arc $PM = \rho\Delta\theta$, and $MQ = \Delta\rho$. The chords PM and PQ, the tangents RPH and TPZ, are drawn; and ZH is drawn perpendicular to PH, Z being any point on the tangent PZ.

When $\Delta s \doteq 0$, the limiting positions of the secants PM and PQ are the tangents RPH and TPZ, respectively; hence,

$$lt\,(\angle\ PMQ) = \angle\ RPK = \pi/2 = \angle\ PHZ,$$
$$lt\,(\angle\ OQP) = \angle\ OPT = \psi = \angle\ HZP,$$

and
$$lt\ \angle\ MPQ = \angle HPZ.$$

Now in a problem of limits the chord of an infinitesimal arc can be substituted for the arc, since the limit of their ratio is unity (Art. 22 and *Cor.*, Art. 46); so

$$lt\frac{\Delta\rho}{\Delta s} = lt\frac{MQ}{\text{chord } PQ} = lt\frac{\sin MPQ}{\sin PMQ};$$

$$\therefore \quad \frac{d\rho}{ds} = \sin HPZ = \frac{HZ}{PZ}. \tag{1}$$

Again, $$lt\frac{\rho\Delta\theta}{\Delta s} = lt\frac{\text{chord } MP}{\text{chord } PQ} = lt\frac{\sin MQP}{\sin PMQ};$$

$$\therefore \quad \frac{\rho d\theta}{ds} = \sin HZP = \frac{HP}{PZ}. \tag{2}$$

From (1) and (2), it follows that, if PZ is taken as ds,

$$ds = PZ, \quad d\rho = HZ, \quad \text{and} \quad \rho\, d\theta = HP.$$

Drawing OT perpendicular to OP, and PA and ON perpendicular to the tangent TP, the length PT is the *polar tangent*; PA, the *polar normal*; OA, the *polar subnormal*; and OT, the *polar subtangent*.

From the right-angled HPZ

$$ds^2 = d\rho^2 + \rho^2\, d\theta^2; \tag{3}$$

$$\sin\psi = \frac{\rho\, d\theta}{ds}, \quad \cos\psi = \frac{d\rho}{ds}, \quad \tan\psi = \frac{\rho\, d\theta}{d\rho}. \tag{4}$$

Polar subt. $= OT = OP\tan\psi = \rho^2\, d\theta/d\rho.$ (5)

Polar subn. $= OA = OP\cot\psi = d\rho/d\theta.$ (6)

Polar tan. $= PT = \sqrt{OP^2 + OT^2} = \rho\sqrt{1 + \rho^2\dfrac{d\theta^2}{d\rho^2}}.$ (7)

Polar norm. $= AP = \sqrt{OP^2 + OA^2} = \sqrt{\rho^2 + \dfrac{d\rho^2}{d\theta^2}}.$ (8)

$$\begin{aligned} p = ON &= OP\sin\psi = \rho^2\, d\theta/ds \\ &= \frac{\rho^2}{\sqrt{\rho^2 + (d\rho/d\theta)^2}}. \end{aligned} \tag{9}$$

$$\phi = \psi + \theta. \tag{10}$$

Corollary. — If PZ represents the velocity at P of a moving

point (ρ, θ) along its path, PK $(= HZ)$ and PH will represent its component velocities at P along the radius vector and a line perpendicular to it.

If the path is a circle with center at 0, ψ is $90°$; and

$$\sin\psi = \sin 90° = 1 = \frac{\rho\, d\theta}{ds}, \text{ from (4), or } \frac{ds}{dt} = \rho\frac{d\theta}{dt}, \quad \therefore \quad v = r\omega;$$

that is, the linear velocity = radius times angular velocity. (See Art. 72.)

EXERCISE X.

1. Find the subtangent, subnormal, tangent, normal, and p of the spiral of Archimedes $\rho = a\theta$.

Ans. subt. $= \rho^2/a$; subn. $= a$; normal $= \sqrt{\rho^2 + a^2}$;
tangent $= \rho\sqrt{1 + \rho^2/a^2}$; $p = \rho^2/(\rho^2 + a^2)^{\frac{1}{2}}$.

2. In the spiral of Archimedes show that $\tan\psi = \theta$; thence find the values of ψ, when $\theta = 2\pi$ and 4π.

Ans. $80°\ 57'$ and $85°\ 27'$.

3. Find the subtangent, subnormal, tangent, and normal of the logarithmic spiral $\rho = a^{\theta}$.

Ans. subt. $= \rho/\log a$; subn. $= \rho\log a$; tan. $= \rho\sqrt{1 + (\log a)^{-2}}$;
norm. $= \rho\sqrt{1 + (\log a)^2}$.

4. Show why the logarithmic spiral is called the *equiangular* spiral, by finding that ψ is constant.

If $a = e$, $\psi = \pi/4$, subt. = subn., and tan. = norm.

5. Find the subtangent, subnormal, and p of the Lemniscate of Bernouilli $\rho^2 = a^2\cos 2\theta$.

Ans. subt. $= -\rho^3/a^2\sin 2\theta$; subn. $= -a^2\sin 2\theta/\rho$;
$p = \rho^3/\sqrt{\rho^4 + a^4\sin^2 2\theta} = \rho^3/a^2$.

6. In the circle $\rho = a\sin\theta$, find ψ and ϕ.

Ans. $\psi = \theta$, and $\phi = 2\theta$.

The angle between two polar curves is found as for the other curves.

7. Find the angle of intersection between the circle $\rho = 2a\cos\theta$, and the cissoid $\rho = 2a\sin\theta\tan\theta$.

Ans. arc tan 2.

CHAPTER V.

MAXIMA AND MINIMA. INFLEXION POINTS.

78. Maxima and Minima. — One of the principal uses of derivatives is to find out under what conditions the value of the function differentiated becomes a maximum or a minimum.

This is often very important in engineering questions, when it is most desirable to know what conditions will make the cost of labor and material a minimum, or will make efficiency and output a maximum.

A maximum value of a function or variable is defined to be a value greater than those values immediately before and after it, and a minimum value to be one less than those immediately before and after it. It follows that the function is increasing before, and decreasing after reaching a maximum value; while it is decreasing before, and increasing after reaching a minimum value.

The points on the graph of $y = f(x)$ at which the function ceases to increase and begins to decrease, or ceases to decrease and begins to increase, are maxima or minima points; and the values of the function at those points are maxima or minima values.

It is to be noted that a maximum value is not necessarily the *greatest* value the function can have nor a minimum the *least*; $f(a)$ is a maximum if it be greater than any other value of $f(x)$ near $f(a)$ and on either side of it; and $f(a)$ is a minimum if it be less than any other value of $f(x)$ near $f(a)$ and on either side of it.

79. The Condition for a Maximum or a Minimum Value. — If $f(x)$ is a function of an increasing variable x; then

for $f(a)$ to be a maximum, $f(x)$ must be increasing just before $f(a)$ and therefore $f'(x)$ must be positive; on the other hand $f(x)$ must be decreasing just after $f(a)$ and therefore $f'(x)$ must be negative. Hence, as x increases through the value a, $f'(x)$ must change from a positive to a negative value. Conversely, if as x increases through the value a, $f'(x)$ changes from a positive to a negative value, $f(a)$ will be a maximum value of $f(x)$.

Hence $f(a)$ will be a maximum value of $f(x)$ if, and only if, $f'(x)$ changes from a positive to a negative value as x increases through the value a.

In the same way it may be seen that $f(a)$ will be a minimum value of $f(x)$ if, and only if, $f'(x)$ changes from a negative to a positive value as x increases through the value a.

This condition has been called the *fundamental condition or test.*

For the cases of most frequent occurrence; when $f(a)$ is a maximum or a minimum, $f'(a) = 0$. In most cases it is a necessary condition for a maximum or a minimum value of a function that the first derivative at that value shall be zero. For in most cases the first derivative $f'(x)$ is *continuous;* and, when continuous, it changes sign by passing through the value *zero only.* But if $f'(x)$ is not continuous, as is the case for some functions, then it may change sign by becoming *infinite* for some finite value of x; for if $f'(x)$ is a fraction whose denominator becomes zero for some finite value of x, $f'(x)$ changes sign as x increases through that value. For example, when $f'(x) = \dfrac{c}{x-2}$, for $x = a = 2$, $f'(a) = \dfrac{c}{2-2} = \infty$; here $f'(x)$ is negative before, and positive after x increases through the value 2; hence, $f(2)$ is a minimum according to the fundamental test. Again, there are exceptional non-algebraic functions for which $f'(x)$, as x increases through some finite value a, changes sign

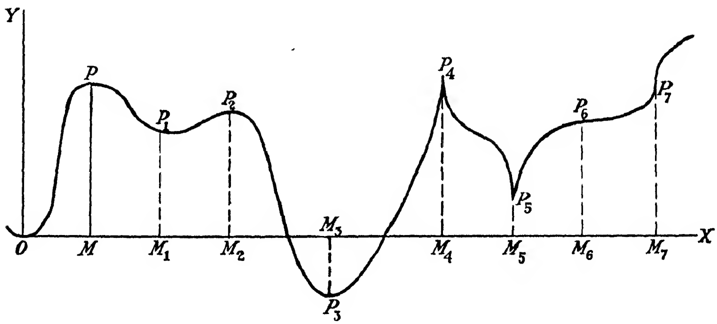
without becoming either zero or infinite. (See Note, Art. 80.)

Excepting such rare functions, a **theorem** may be stated thus:

For all algebraic functions any value of x which makes $f(x)$ a maximum or a minimum is a root of $f'(x) = 0$ or $f'(x) = \infty$.

The converse of this theorem is not true; that is, any root of $f'(x) = 0$ or $f'(x) = \infty$ does not necessarily make $f(x)$ either a maximum or a minimum. These roots are called *critical* values of x, and each root may be tested by rule.

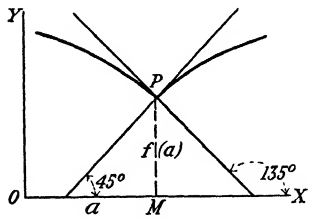
80. Graphical Illustration. — Let $P \ldots P_3 \ldots P_7$ be the locus of $y = f(x)$. Then $f(x)$ will be represented by the ordinate of the point (x, y), and $f'(x)$ by the slope of the locus at the point (x, y). By definition, the ordinates MP, M_2P_2, and M_4P_4 represent maxima of $f(x)$; while O, M_1P_1, M_3P_3, and M_5P_5 represent minima. (Art. 78.)



The slope $f'(x)$ is positive immediately before a maximum ordinate, and negative immediately after; while the slope is negative immediately before a minimum ordinate and positive after. The slope $f'(x)$ is 0 or ∞ at any point whose ordinate $f(x)$ is either a maximum or a minimum. The slope $f'(x)$ is *discontinuous* at the points P_4 and P_5, where it changes sign by becoming infinite as x increases through the values OM_4 and OM_5; that is, $f'(x) = \infty$.

The slope $f'(x)$ is 0 at P_6 and ∞ at P_7; but it does not change sign at either point, and neither M_6P_6 nor M_7P_7 is a maximum or a minimum *ordinate;* it does, however, change in *value* at each point, P_6 being a point where the *slope* $f'(x)$ is a minimum and P_7 one where it is a maximum. The points P_6 and P_7 are *inflexion* points, at which the curve changes from being concave downward to upward, or *vice versa.*

Note. — Points such as P_4 and P_5 occur on railroad "Y's," and such points where branches of a curve end tangent to each other are called *cusps.* At a point on a non-algebraic curve where branches end and are not tangent to each other, called a *shooting point,* $f'(x)$ may change abruptly from a positive finite value to a negative value, or *vice versa;* hence,



$f(a)$ would be a maximum or a minimum without $f'(x)$ becoming either zero or infinite. The supplementary figure shows a shooting point at which $f(a)$ is a minimum; $f'(x)$ becoming -1 as x increases to a, and $+1$ as x decreases to the same value a, thus changing from a negative to a positive value as x increases through the value a.

It is to be noted that, while on an exceptional curve like the one shown the tangents at a maximum or a minimum point may have various directions, on any algebraic curve the tangent is parallel to one or other of the two rectangular axes; that is, the tangent at a maximum or a minimum point is *horizontal,* the slope being continuous; otherwise it is *vertical;* and on only exceptional non-algebraic curves will it have any other direction.

It may be noted also, as in the graphical illustration given, that maxima and minima occur alternately; that is, a minimum between any two consecutive maxima and *vice versa.* It may be seen that a maximum may be less than

some minimum not consecutive, since by definition it is *necessarily* greater than those values only *immediately* before and after it. It may be seen also that when the **slope** is continuous at least one inflexion point must occur between a maximum and a minimum point. The only inflexion points *marked* on the curve are P_6 and P_7, occurring where $f'(x) = 0$ and ∞, but $f'(x)$ may have any value at an inflexion point, although its rate, $f''(x)$, must change sign there, becoming 0 or ∞. Hence, at *any* inflexion point, a point where the *slope* $f'(x)$ is a maximum or a minimum, $f''(x) = 0$ or ∞. The converse is not true, for $f''(x)$ may be 0 or ∞ at other points.

81. Rule for Applying Fundamental Test. — *Let a be a critical value given by either $f'(x) = 0$ or $f'(x) = \infty$, or, in general, any value of x to be tested, and Δx a small positive number; then:*

If $f'(a - \Delta x)$ is positive and $f'(a + \Delta x)$ is negative,

$f(a)$ is a maximum of $f(x)$. (Art. 79.)

If $f'(a - \Delta x)$ is negative and $f'(a + \Delta x)$ is positive,

$f(a)$ is a minimum of $f(x)$. (Art. 79.)

If $f'(a - \Delta x)$ and $f(a + \Delta x)$ are both positive or both negative, $f(a)$ is neither a maximum nor a minimum of $f(x)$.

This rule is general and is valid for all functions that are continuous one-valued functions, which comprise all those usually encountered in this connection.

82. While the **rule** just stated applies in every case; when $f'(x)$, as well as $f(x)$, is continuous and therefore the critical values of x are roots of $f'(x) = 0$, a rule usually easier to apply may be deduced from the fundamental test or condition.

Let a be a critical value of x given by $f'(x) = 0$. If $f(a)$ is a maximum value of $f(x)$, $f'(x)$ changes from a positive to a negative value as x increases through a; therefore, near a, $f'(x)$ is a decreasing function, and therefore, *its* derivative,

$f''(x)$, must be negative near a. But if $f''(a)$ is not zero, then near a the sign of $f''(x)$ is that of $f''(a)$. Hence $f''(a)$, if it is not zero, will be negative when $f(a)$ is a maximum value of $f(x)$.

In the same way it is seen that $f''(a)$, if it is not zero, will be positive when $f(a)$ is a minimum value of $f(x)$.

Conversely, $f(a)$ will be a maximum or a minimum value of $f(x)$ according as $f''(a)$ is negative or positive.

Hence this **rule** for determining the maxima and minima values of $f(x)$ when $f(x)$, $f'(x)$ are continuous:

The roots of the equation $f'(x) = 0$ are, in most cases, the values of x which make $f(x)$ a maximum or a minimum.

If a be a root of $f'(x) = 0$; then $f(a)$ will be a maximum value of $f(x)$, if $f''(a)$ is negative, but a minimum, if $f''(a)$ is positive.

Illustrative Examples.

Example 1. Examine the function $(x + 2)^2(x - 1)^3$ for maximum and minimum values.

$$\begin{aligned} f(x) &= (x + 2)^2(x - 1)^3, \\ f'(x) &= 2(x + 2)(x - 1)^3 + 3(x + 2)^2(x - 1)^2, \\ &= 5(x + 2)(x - 1)^2(x + 4/5). \end{aligned}$$

Putting $(x + 2)(x - 1)^2(x + 4/5) = 0$, the critical values are found to be the roots, $x = -2, +1, -4/5$, which are to be examined. The value $x = -2$ may be tested by the original function; thus

$$\begin{aligned} f(-2) &= (-2 + 2)^2(-2 - 1)^3 = 0, \\ f(-2 - \Delta x) &= \overline{\Delta x}^2(-2 - \Delta x - 1)^3 < 0, \\ f(-2 + \Delta x) &= \overline{\Delta x}^2(-2 + \Delta x - 1)^3 < 0. \end{aligned}$$

Although the function is 0 for $x = -2$, it is greater than the values just before and after, since the values gotten by taking x a little before and a little after -2 come out negative.

It follows that for $x = -2$ the function has a maximum value, which is 0.

In the same way the value $x = 1$ may be tested; thus

$$f(1) = 0, \qquad f(1 - \Delta x) = -(1 - \Delta x + 2)^2 \overline{\Delta x}^3 < 0,$$
$$f(1 + \Delta x) = (1 + \Delta x + 2)^2 \overline{\Delta x}^3 > 0.$$

Although $f'(x) = 0$ for $x = 1$, the function is neither a maximum nor a minimum for $x = 1$; since $f(1) = 0$, while greater than the value before, is not greater than the value just after. For the critical value $x = -4/5$, apply the *fundamental test*; thus, for $x = -4/5 - \Delta x$, $f'(x)$ is negative, and for $x = -5/4 + \Delta x$, $f'(x)$ is positive. Since $f'(x)$ changes from a negative to a positive value as x increases through the value $-4/5$, the function has a minimum value for that value of x.

Example 2.

$$f(x) = x^3 - 4x^2 + 5x - 2.$$
$$f'(x) = 3x^2 - 8x + 5 = (3x - 5)(x - 1),$$
$$f''(x) = 6x - 8.$$

Putting $f'(x) = (3x - 5)(x - 1) = 0$, the roots $x = 5/3$ and $x = 1$ are found to be the critical values, which in $f''(x)$ give

$$f''(5/3) = 6 \times 5/3 - 8 = 2, \qquad f''(1) = 6 - 8 = -2.$$

Since $f''(x)$ is positive for $x = 5/3$, $f(5/3) = -4$ is a minimum value of the function; and since $f''(x)$ is negative for $x = 1$, $f(1) = 0$ is a maximum value of the function.

Example 3.

$$f(x) = (x + 1)^{2/3}(x - 5)^2.$$
$$\begin{aligned} f'(x) &= 2(x + 1)^{2/3}(x - 5) + 2/3\,(x + 1)^{-1/3}(x - 5)^2 \\ &= (x + 1)^{-1/3}(x - 5)[2(x + 1) + 2/3\,(x - 5)] \\ &= \frac{x - 5}{(x + 1)^{1/3}}(8/3\,x - 4/3). \end{aligned}$$

Putting $f'(x) = 0$ gives the roots $x = 5$ and $x = 1/2$, and putting $f'(x) = \infty$ gives the root $x = -1$. Since $f(5 - \Delta x) > 0$ and $f(5 + \Delta x) > 0$, $f(5) = 0$ is a minimum, and since $f(-1 - \Delta x) > 0$ and $f(-1 + \Delta x) > 0$, $f(-1) = 0$ is a minimum. Since $f'(\frac{1}{2} - \Delta x)$ is positive and $f'(\frac{1}{2} + \Delta x)$ is negative, $f(\frac{1}{2})$ is a maximum.

EXERCISE X_2.

Examine the following functions for maxima or minima. After finding the critical values of the variable, use that method which appears most applicable. When the second derivative is readily found, its use is advisable; otherwise apply the fundamental test, or test by substitution in the original function.

1. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 12.$
2. $f(x) = (x - 4)^2(x - 3)^2.$
3. $f(x) = x^4 - 6x^2 + 10.$
4. $f(x) = x + 4/x$
5. $f(x) = x(x^2 - 1).$
6. $f(x) = x^2(x - 1).$
7. $f(x) = x^2/2 + 1/x.$
8. $f(x) = x/\sqrt{x - 1}.$
9. $f(\theta) = 2 \tan \theta + \sec^2 \theta.$
10. $f(x) = e^x + e^{-x}.$
11. $f(x) = x - e^x.$
12. $f(x) = \dfrac{x + 2}{x} \sqrt{1 + x^2}.$
13. $f(x) = \dfrac{(a - x)^3}{a - 2x}.$
14. $f(x) = (x - 1)^4(x + 2)^3.$
15. $f(x) = (x - 2)^5(2x + 1)^4.$

83. While the above rule * (Art. 82) is all that is needed in most cases, it does not provide for the case when the critical value a makes $f''(x)$ become zero. When $f''(a) = 0$, $f(a)$ may be *either* a maximum or a minimum, or it may be *neither*, and the point on the graph of $f(x)$ may, or may not, be a point of inflexion; so an extension of the rule is needed to provide for cases where $f''(x)$ and the succeeding derivatives may in turn become zero for the value, a.

If no derivative is found that does not become zero when a is substituted for x, then recourse may be had to the fundamental test, that rule applying in every case. But if $f'(a)$, $f''(a), \ldots, f^{n-1}(a)$ all are found to be zero, and $f^n(a)$ not zero; then the following **rule**, inclusive of the preceding, applies. *Let a be a critical value of x given by $f'(x) = 0$, and let a be substituted for x in the successive derivatives of $f(x)$.*

If the order n of the first of the derivatives that is not zero is an ***even*** *integer, $f(a)$ will be a maximum or a minimum of $f(x)$ according as this derivative is negative or positive.*

If the order n of the first of the derivatives that is not zero is

* Page 118, Art. 82.

an **odd** *integer, $f(a)$ will be neither a maximum nor a minimum of $f(x)$ regardless of the sign of this derivative.*

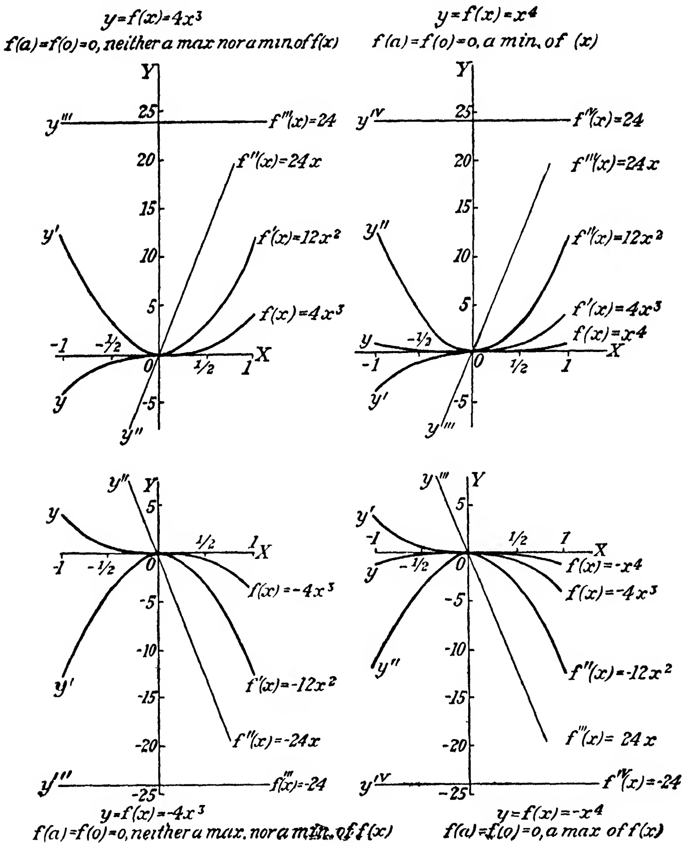
Note. — This conclusion can be deduced by examining the signs of the derivatives near a; thus, as follows:

If $f'(a)$ and $f''(a)$ be zero but $f'''(a)$ not zero; since $f'''(x)$, the rate of $f''(x)$, has when x is a a value not zero; $f''(x)$, the rate of $f'(x)$, is then increasing or decreasing according as $f'''(a)$ is positive or negative, and, since it is zero when x is a, it must change sign as x increases through a; therefore, $f'(x)$, the rate of $f(x)$, must be either decreasing before *and* increasing after, *or* increasing before *and* decreasing after x is a, and so, continuing to be positive or negative according as $f'''(a)$ is positive or negative, does *not* change sign as x increases through a; hence $f(a)$ is neither a maximum nor a minimum of $f(x)$ *regardless* of the sign of $f'''(a)$.

Now if $f'''(a)$ also is zero but $f^{\text{IV}}(a)$ not zero; since $f^{\text{IV}}(x)$, the rate of $f'''(x)$, has when x is a a value not zero, $f'''(x)$, the rate of $f''(x)$, is then decreasing *or* increasing according as $f^{\text{IV}}(x)$ is *negative* or *positive* and, since it is zero when x is a, it must change sign as x increases through a; therefore, $f''(x)$, the rate of $f'(x)$, must be either increasing before *and* decreasing after, *or* decreasing before *and* increasing after x is a, and so, continuing negative or positive according as $f^{\text{IV}}(a)$ is negative or positive, does *not* change sign as x increases through a; $f'(x)$, the rate of $f(x)$, must then be either decreasing, or increasing before *and* after x is a, and, as it is zero when x is a, it changes from a positive to a negative value, *or* from a negative to a positive value, as x increases through a, according as $f^{\text{IV}}(a)$ is negative or positive; hence $f(a)$ is a maximum or a minimum of $f(x)$ according as $f^{\text{IV}}(a)$, the first of the derivatives that is not zero, is *negative* or *positive.*

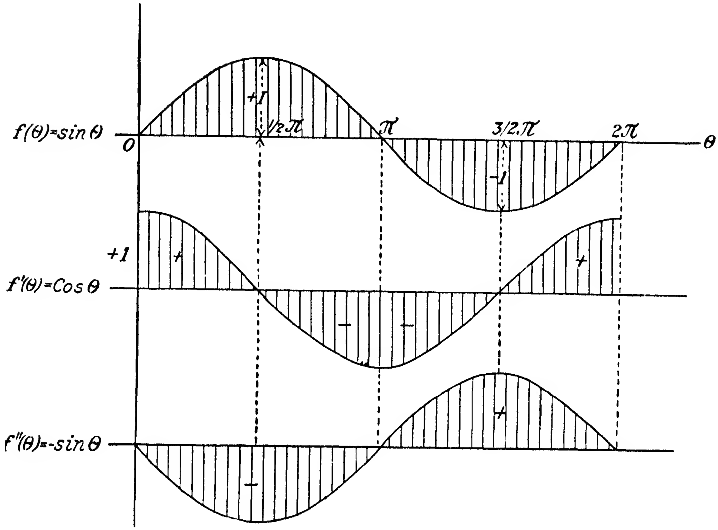
In the same way it follows that, if $f^{\text{V}}(a)$ is the first of the derivatives that is not zero, $f(a)$ is neither a maximum nor a minimum of $f(x)$ *regardless* of the sign of $f^{\text{V}}(a)$; and that,

if $f^{\text{VI}}(a)$ is the first, $f(a)$ is a maximum or a minimum of $f(x)$ according as $f^{\text{VI}}(a)$ is *negative* or *positive;* and so on for the succeeding derivatives: hence the inclusive ***rule*** given. (For proof by Taylor's Theorem, see Art. 158.)



84. Typical Illustrations. — The foregoing deductions may be verified by the graphs of the successive derivatives of x^3, $-x^3$, x^4, and $-x^4$, referred to the same axes as those of the graphs of the functions. The usual case when $f'(a)$ is

zero and $f''(a)$ not zero, is well illustrated by the graphs of the function $\sin\theta$ and its derivatives.



$$f(\theta) = \sin\theta;\ \text{where } \theta \text{ is in radians.}$$
$$f'(\theta) = \cos\theta = 0; \quad \therefore \quad \theta = \pi/2,\ \tfrac{3}{2}\pi,\ \ldots,\ (\pi/2 + n\pi);$$
$$f''(\theta) = -\sin\theta, \quad f''(\pi/2) = -;$$
$$\therefore \quad f(\pi/2) = 1 \text{ is a maximum value of } f(x).$$

$$f''(\tfrac{3}{2}\pi) = +; \quad \therefore \quad f(\tfrac{3}{2}\pi) = -1 \text{ is a minimum value of } f(x).$$
$$f''(\theta) = -\sin\theta = 0; \quad \therefore \quad \theta = 0, \pi, \ldots, n\pi;$$
$$\therefore \quad (0, 0), (\pi, 0), \ldots \text{ are points of inflexion.}$$

$$f'''(\theta) = -\cos\theta;\ f'''(o) = -;$$
$$\therefore \quad f'(o) = 1 \text{ is a maximum slope.}$$

$$f'''(\pi) = +; \quad \therefore \quad f'(\pi) = -1 \text{ is a minimum slope.}$$

The graphs make manifest that for a maximum or a minimum value of the function the first derivative passes through zero; being + before and − after for a max., and − before and

+ after for a min.; and that hence the second derivative is − for a max. and + for a min.; also that at an inflexion point the second derivative is zero. The graph of sin θ makes manifest at once the maxima and minima values of the function and the value of the angle in radians that makes the function a maximum or a minimum.

The graph of the first derivative shows that the first derivative of any continuous function when continuous itself, changes sign by passing through zero only, for the ordinate changes sign by becoming zero only, as the graph crosses the axis of θ; and it shows that in passing through zero the ordinate changes from plus to minus, *or* from minus to plus according as the abscissa of the point of crossing corresponds to a maximum or a minimum ordinate of the graph of the function; and that as it crosses the axis the ordinate is either decreasing or increasing. The graph of the second derivative shows by the direction or sign of its ordinate *at* or *near* the point where the graph of the first derivative crosses the axis whether the first derivative is decreasing or increasing as it passes through zero at that point, the sign being minus or plus according as the first derivative is decreasing or increasing; that is, according as the abscissa of the point corresponds to a maximum or a minimum ordinate of the graph of the function.

Auxiliary Theorems. — By use of the following theorems, which are obvious, the solutions of problems in maxima and minima are often simplified:

(i) Any value of x which makes $c + f(x)$ a maximum or a minimum makes $f(x)$ a maximum or a minimum; and conversely.

(ii) Any value of x which makes $c \cdot f(x)$, c being positive, a maximum or a minimum makes $f(x)$ a maximum or a minimum; and conversely. If c is negative and $f(a)$ is a maximum, $c \cdot f(a)$ is a minimum.

(iii) Any value of x which makes $f(x)$ positive, and a

maximum or a minimum, makes $[f(x)]^n$ a maximum or a minimum, n being any positive whole number.

(iv) Since $f(x)$ and $\log f(x)$ increase and decrease together, any value which makes $f(x)$ a maximum or a minimum makes $\log f(x)$ a maximum or a minimum; and conversely.

(v) Since when $f(x)$ increases its reciprocal decreases, any value of x which makes $f(x)$ a maximum or a minimum makes its reciprocal a minimum or a maximum.

EXERCISE XI.

Examine $f(x)$ for maxima and minima when:

1. $f(x) = x^5 - 5x^4 + 5x^3 - 1.$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3)$

$= 5x^2(x-1)(x-3) = 0; \quad \therefore \quad x = 0, 1, 3.$

$f''(x) = 20x^3 - 60x^2 + 30x; \; f'''(x) = 60x^2 - 120x + 30.$

$f''(0) = 0, \; f'''(0) = 30; \quad \therefore \quad f(0) = -1$

is neither a max. nor a min.

$f''(1) = 20 - 60 + 30 = -10; \quad \therefore \quad f(1) = 0$ is a max.

$f''(3) = 540 - 540 + 90 = 90; \quad \therefore \quad f(3) = -28$ is a min.

By plotting the graph of $f(x)$ these results may be verified.

2. $f(x) = x^3 - 3x^2 + 3x + 7.$

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$

$= 3(x-1)^2 = 0; \quad \therefore \quad x = 1, 1.$

$f''(x) = 6x - 6 = 6(x-1); \; f'''(x) = 6.$

$f''(1) = 0; f'''(1) = 6; \quad \therefore \quad f(1) = 8$ is neither a max. nor a min.

3. $f(x) = 3x^4 - 4x^3 + 1.$

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1) = 0; \quad \therefore \quad x = 0, 1.$

$f''(x) = 36x^2 - 24x = 12x(3x - 2); \; f'''(x) = 72x - 24.$

$f''(0) = 0; \; f'''(0) = -24; \quad \therefore \quad f(0) = 1$

is neither a max. nor a min.

$f''(1) = 12; \quad \therefore \quad f(1) = 0$ is a min.

4. $f(x) = 3x^5 - 125x^3 + 2160x.$

$f'(x) = 15x^4 - 375x^2 + 2160 = 15(x^4 - 25x^2 + 144) = 0;$

$\therefore \quad x = \pm 3, \pm 4.$

$f''(x) = 15(4x^3 - 50x); \; f''(3)$ is neg.; $\quad \therefore \quad f(3)$ is max.

$f''(4)$ is positive; $\therefore f(4)$ is min.; $f''(-3)$ is positive; $\therefore f(-3)$ is min.; $f''(-4)$ is negative; $\quad \therefore \quad f(-4)$ is max.

5. $f(x) = x^3 - 3x^2 + 6x + 7.$

$$f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 0; \quad \therefore \quad x = 1 \pm \sqrt{-1}.$$

Hence no *real* value of x makes $f(x)$ a max. or a min.

6. Examine $c + \sqrt{4a^2x^2 - 2ax^3}$ for max. and min.

Let $f(x) = 2ax^2 - x^3$ (by Art. 87).

$$f'(x) = 4ax - 3x^2 = 4(4a - 3x) = 0; \quad \therefore x = 4/3\,a,\ 0.$$

$$f''(x) = 4a - 6x;\ f''(0) = 4a;\ \therefore f(0) = c \text{ is a min.}$$

$$f''(4/3\,a) = 4a - 8a = -4a;$$

$$\therefore \quad f(4/3\,a) = c + 8a^2\sqrt{3}/9 \text{ is a max.}$$

7. $y = a - b(x-c)^{\frac{2}{3}}$, and $y = a - b(x-c)^{\frac{1}{3}}$.

$$\frac{dy}{dx} = -\frac{2b}{3(x-c)^{\frac{1}{3}}} = \infty; \quad \therefore \quad x = c, \text{ and it can be seen that } \frac{dy}{dx}$$

changes from $+$ to $-$ when x passes through the value c, hence $f(c) = a$ is a max.

When $f(x) = a - b(x-c)^{\frac{1}{3}}$; $\dfrac{dy}{dx} = -\dfrac{b}{3(x-c)^{\frac{2}{3}}} = \infty; \quad \therefore \quad x = c;$

here it can be seen that $f'(x)$ or $\dfrac{dy}{dx}$ does not change sign as x passes through c, and, therefore, the function has neither max. nor min.

8. Examine $(x-1)^4(x+2)^3$ for max. and min.

$$f'(x) = (x-1)^3(x+2)^2(7x+5) = 0; \quad \therefore \quad x = 1, -2, -\tfrac{5}{7}.$$

$$f'(1 - \Delta x) \text{ is } -,\ f'(1 + \Delta x) \text{ is } +; \quad \therefore \quad f(1) = 0 \text{ is a min.}$$

$$f'(-\tfrac{5}{7} - \Delta x) \text{ is } +,\ f'(-\tfrac{5}{7} + \Delta x) \text{ is } -; \quad \therefore f(-\tfrac{5}{7}) \text{ is a max.}$$

$f'(-2 - \Delta x)$ and $f'(-2 + \Delta x)$ are both $+$; hence $f(-2)$ is neither max. nor min.

9. Examine $\dfrac{(a-x)^3}{a - 2x}$ for max. and min.

$f'(x) = (a-x)^2(4x - a)/(a - 2x)^2$; $f'(x) = 0$ gives $x = a, a/4$;

$f'(x) = \infty$ gives $(a - 2x)^2 = 0$, or $x = a/2$.

$f'(a/4)$ changes from $-$ to $+$ as x passes through $a/4$; $\therefore f(a/4)$ is min. When $x = a$, or $a/2$, $f'(x)$ does not change sign; $\therefore f(a)$ and $f(a/2)$ are neither max. nor min.

10. When $f(x) = (x-1)(x-2)(x-3)$, $f(2 - 1/\sqrt{3}) = \frac{2}{9}\sqrt{3}$ is a max., and $f(2 + 1/\sqrt{3}) = -\frac{2}{9}\sqrt{3}$ is a min.

11. Show that the maximum value of $\sin\theta + \cos\theta$ is $\sqrt{2}$.

12. Show that the maximum value of $a\sin\theta + b\cos\theta$ is $\sqrt{a^2 + b^2}$.

13. Show that e is a minimum of $x/\log x$.

14. Show that $1/ne$ is a maximum of $\log x/x^n$.

15. Show that $e^{1/e}$ is a maximum of $x^{1/x}$.

16. Show that 1 is a maximum of $2\tan\theta - \tan^2\theta$.

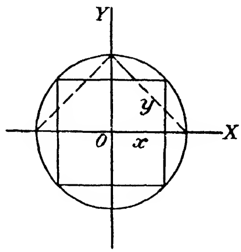
17. Find the maximum value of $\tan^{-1}x - \tan^{-1}x/4$, the angles being taken in the first quadrant. *Ans.* $\tan^{-1}\frac{3}{4}$.

18. Show that 2 is a maximum ordinate and -26 is a minimum ordinate of the curve $y = x^5 - 5x^4 + 5x^3 + 1$.

PROBLEMS IN MAXIMA AND MINIMA.

1. Find the maximum rectangle that can be inscribed in a circle of radius a. Let $2x$ = base and $2y$ = altitude; then

$$\text{area } A = 4xy = 4x\sqrt{a^2 - x^2}.$$



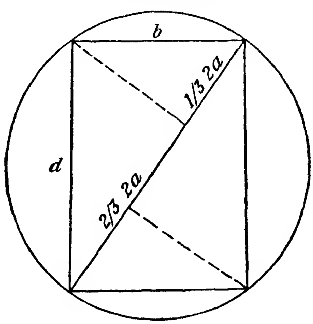
Take
$$f(x) = x^2(a^2 - x^2) = a^2x^2 - x^4 \text{ [by Art. 87];}$$
$$f'(x) = 2a^2x - 4x^3 = 2x(a^2 - 2x^2) = 0; \quad \therefore \quad x = 0,\ a/\sqrt{2};$$
$$f''(x) = 2a^2 - 12x^2;\ f''(0) = 2a^2; \quad \therefore \quad f(0) = 0 \text{ is min.}$$
$$f''(a/\sqrt{2}) = 2a^2 - 6a^2 = -4a^2; \quad \therefore \quad f(a/\sqrt{2}) = a^4/4.$$
$$\therefore \quad A = 4\sqrt{a^4/4} = 2a^2$$

is the area of the maximum rectangle, which is a square.

Note. — By Geometry without the Calculus method:

$$A = 2ay = 2a\sqrt{a^2 - x^2}\,\Big]_{x=0} = 2a^2,$$

since the radical quantity is evidently greatest when $x = 0$.



2. The strength of a beam of rectangular cross section varies as the breadth b and as d^2, the square of the depth. Find the dimensions of the section of the strongest beam that can be cut from a cylindrical

log whose diameter is $2a$. Strength $\propto bd^2$; $\therefore$ strength $= kbd^2$, where k is a constant; let

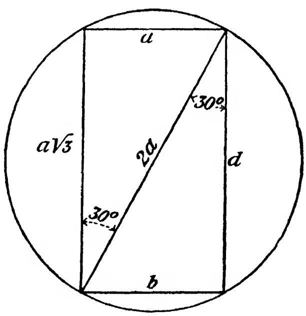
$$f(b) = b(4a^2 - b^2) = 4a^2b - b^3;$$

$$f'(b) = 4a^2 - 3b^2 = 0; \quad \therefore \quad b = \frac{2a}{\sqrt{3}}, \quad d = \sqrt{\frac{2}{3}}(2a).$$

$$f(b) = \frac{2a}{\sqrt{3}}\left(4a^2 - \frac{4a^2}{3}\right) = \frac{2a}{\sqrt{3}} \cdot \frac{2}{3}(4a^2) = \frac{16a^3}{3\sqrt{3}}.$$

Hence, the rectangle may be laid off on the end of the log by drawing a diameter and dividing it into three equal parts; from the points of division drawing perpendiculars in opposite directions to the circumference and joining the points of intersection with the ends of the diameter, as in the figure. The strength of the beam is about 0.65 of that of the log, but it is the strongest beam of rectangular section.

3. The stiffness of a rectangular beam varies as the breadth b and as d^3, the cube of the depth. Find the dimensions of the stiffest beam that can be cut from the log.



Stiffness $\propto bd^3$; $\therefore$ stiffness $= kbd^3$ (k constant); let stiffness $= b(4a^2 - b^2)^{\frac{3}{2}}$. Take

$$f(b) = 4a^2b^{\frac{2}{3}} - b^{\frac{8}{3}}; \; f'(b) = \tfrac{8}{3}(a^2b^{-\frac{1}{3}} - b^{\frac{5}{3}}) = 0;$$

$$\therefore \quad b^2 = a^2 \quad \text{or} \quad b = a; \quad \therefore \quad d = (4a^2 - a^2)^{\frac{1}{2}} = a\sqrt{3}.$$

To draw the rectangle, lay off from ends of a diameter chords at angles of 30° with diameter and join ends of chords with ends of diameter.

4. A square piece of pasteboard is to be made into a box by cutting out a square at each corner. Find the side of the square cut out, so that the remainder of the sheet will form a vessel of maximum capacity. Let a be side of square sheet and x side of square cut out; then

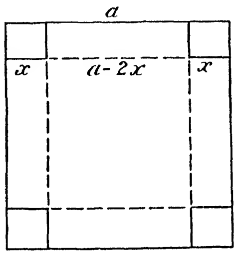
$f(x) = x(a - 2x)^2.$

$f'(x) = -4x(a - 2x) + (a - 2x)^2 = (a - 6x)(a - 2x) = 0;$

$$\therefore \quad a = 1/6\,a,\ 1/2\,a.$$

$f(1/6\,a) = 1/6\,a\,(a - 1/3\,a)^2 = 2/27\,a^3$, maximum capacity.

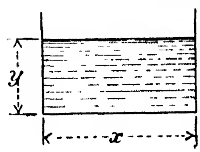
$f(1/2\,a) = 1/2\,a\,(a - a) = 0$, minimum.



5. A rectangular sheet of tin $15'' \times 8''$ has a square cut out at each corner. Find the side of the square so that the remainder of the sheet may form a box of maximum contents.

Ans. $1\frac{2}{3}''$.

6. A channel rectangular in section, carrying a given volume of water, is to be so proportioned as to have a minimum wetted perimeter. Find the proportions of the channel.



Let x be the width of the bottom, and y the height of the water surface. Since the given volume is proportional to the cross section,

$$xy = V, \text{ where } V \text{ is constant.} \tag{1}$$

$$p = x + 2y = x + \frac{2V}{x}, \text{ from (1);}$$

$$\frac{dp}{dx} = 1 - \frac{2V}{x^2} = 0. \quad \therefore \quad x = \sqrt{2V} = \sqrt{2xy};$$

that is,

$$2V = x^2 = 2xy, \text{ or } x = 2y.$$

To show that this makes p a minimum; note that for $x^2 < 2V$, $\frac{dp}{dx}$ is negative, and for $x^2 > 2V$, $\frac{dp}{dx}$ is positive, therefore for $x^2 = 2V$, or $x = 2y$, p is a minimum.

7. Find the dimensions of a conical tent that for a given volume will require the least cloth.

$V = \frac{1}{3}\pi r^2 h$; $\therefore$ $h = 3V/\pi r^2$, where r is radius of base and h altitude. (1)

$$S = \pi r\sqrt{r^2+h^2} = \pi r(r^2 + 9V^2/\pi^2 r^4)^{\frac{1}{2}} = (\pi^2 r^4 + 9V^2/r^2)^{\frac{1}{2}},$$

S denoting lateral surface; let

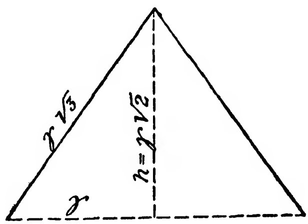
$$f(r) = \pi^2 r^4 + 9V^2/r^2 \text{ (by Art. 87),}$$

$$f'(r) = 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0, \quad \therefore \quad r = \frac{\sqrt{2}}{2}\sqrt[3]{\frac{6V}{\pi}};$$

$$f''(r) = 12\pi^2 r^2 + 54V^2/r^4, \text{ positive for any } r;$$

hence $$r = \frac{\sqrt{2}}{2}\sqrt[3]{\frac{6V}{\pi}} \text{ makes } S \text{ a minimum.}$$

From (1), $h = \sqrt[3]{\frac{6V}{\pi}} = r\sqrt{2}$; and slant height $= r\sqrt{3}$.



8. From a given circular sheet of tin, find the sector to be cut out so that the remainder may form a conical vessel of maximum capacity.

Ans. Angle of sector $= (1 - \sqrt{\frac{2}{3}})\, 2\pi = 66° 14'$.

9. The work of propelling a steamer through the water varies as the cube of her speed; show that her most economical rate per hour against a current running n miles per hour is $3n/2$ miles per hour.

Let v = speed of the steamer in miles per hour;
then cv^3 = work per hour, c being a constant;
and $v - n$ = the actual distance advanced per hour.
Hence, $cv^3/v - n$ = the work per mile of actual advance.

Find the most economical speed against a current of 4 miles per hour.

10. The cost of fuel consumed by a steamer varies as the cube of her speed, and is $25.00 per hour when the speed is 10 miles per hour. The other expenses are $100 per hour. Find the most economical speed. Let C = cost per hour for fuel at speed of v miles per hour;

then $$C : \$25 = v^3 : (10)^3; \quad \therefore \quad C = \$25\,v^3/(10)^3;$$

$$f(v) = \frac{\$25\,v^3}{(10)^3}\cdot\frac{d}{v} + \frac{\$100\,d}{v}; \quad \text{where } d \text{ is distance and } \frac{d}{v} \text{ is hours.}$$

$$f'(v) = \frac{50}{(10)^3}vd - \frac{100\,d}{v^2} = 0;$$

$$v^3 = 2000, \quad \text{or} \quad v = \sqrt[3]{2000} = 12.6 \text{ miles per hr.}$$

$$f(12.6) = \$12\,d \text{ (approximately)};$$

hence cost for one hour about \$150; cost for running 10 miles at 12.6 miles per hour about \$120, while the cost for running the 10 miles at 10 miles per hour is \$125, and at 15 miles per hour the cost for running 10 miles is about \$123.

11. The amount of fuel consumed by a steamer varies as the cube of her speed. When her speed is 15 miles per hour she burns $4\frac{1}{2}$ tons of coal per hour at \$4.00 per ton. The other expenses are \$12.00 per hour. Find her most economical speed and the minimum cost of a voyage of 2080 miles. *Ans.* 10.4 mi. per hr.; \$3600.

12. A vessel is anchored 3 miles off shore. Opposite a point 5 miles farther along the shore, another vessel is anchored 9 miles from the shore. A boat from the first vessel is to land a passenger on the shore and then proceed to the other vessel. Find the shortest course of the boat.

Let h_1 be the distance the boat goes from first vessel to shore and h_2 the distance from the shore to the other vessel; then

$$f(x) = h_1 + h_2 = (3^2 + x^2)^{\frac{1}{2}} + [9^2 + (5 - x)]^{\frac{1}{2}};$$

$$f'(x) = \frac{x}{(9 + x^2)^{\frac{1}{2}}} - \frac{5 - x}{(81 + (5 - x)^2)^{\frac{1}{2}}} = 0;$$

whence $$x = \pm\tfrac{5}{4}; \quad h_1 + h_2 = \tfrac{13}{4} + \tfrac{39}{4} = 13 \text{ miles.}$$

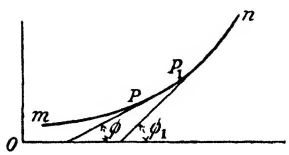
13. Find the number of equal parts into which a given number n must be divided that their continued product may be a maximum. Let

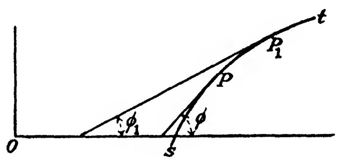
$$f(x) = \left(\frac{n}{x}\right)^x; \quad f'(x) = \left(\frac{n}{x}\right)\left(\log\frac{n}{x} - 1\right) = 0;$$

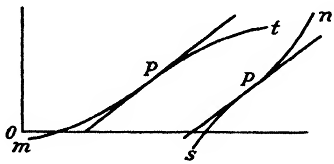
$$\therefore \quad \log\frac{n}{x} = 1; \quad \frac{n}{x} = e, \quad \text{and} \quad x = \frac{n}{e};$$

hence the number of parts is n/e, and each part is e.

85. Inflexion Points. — Where the *slope* of the graph of the function is a maximum or a minimum is at the points where the ordinate of the first derivative has its greatest positive or negative value, and those points are precisely where the ordinate of the second derivative is zero, those values of θ that make the *slope* of the function a maximum or a minimum being those that make *its* derivative, the *second*, zero. These points are inflexion points on the graph of the function, where the curve changes from being concave upward to downward, or downward to upward. When a curve as *mn* is concave upward its slope evidently increases as the abscissa of the generating point increases; hence its derivative, the second (the flexion), is positive. When a curve as *st* is concave downward its slope evidently decreases as the abscissa increases; hence the second derivative is negative. At a point of inflexion, as P on *mt* or *sn*, the tangent crosses the curve at the point of contact, and on opposite





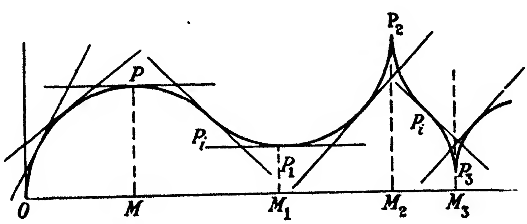


sides the curve is concave in opposite directions, therefore, the second derivative has opposite signs. Hence, at a point of inflexion the first derivative, the slope of the curve, has a maximum or a minimum value. To test a curve for points of inflexion is to test its slope for maxima and minima. In the case of roads or paths that change direction of curvature

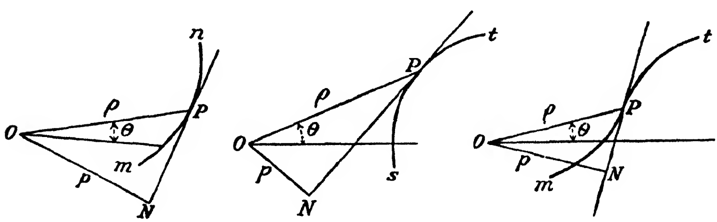
in a horizontal plane the inflexion point is usually called the point of reverse curvature, and when the curved grade changes direction of curvature in a vertical plane the inflexion point is where the grade is greatest or least on that part of the road. The roots of the second derivative $= 0$ or ∞ are the critical values to be tested for points of inflexion, and the sign of the third derivative, when the critical value is given by the second derivative $= 0$, indicates whether the second derivative, the flexion, is decreasing or increasing as the critical value is passed, the sign being minus or plus according as the second derivative is decreasing or increasing; that is, according as at the critical value the slope is a maximum or a minimum, or as the curve is concave upward before and downward after or the reverse. These conclusions are verified by the graphs of $\sin\theta$ and its successive derivatives.

When the critical value is given by the second derivative $= \infty$, or, in general, when any value is to be tested, the fundamental test may be applied to determine the sign of the second derivative before and after the value to be tested, and thus to determine the concavity and existence or non-existence of inflexion.

It is evident that at a maximum point on a curve the curve is concave downward both sides of the horizontal tangent point and concave upward both sides of a vertical tangent point, while the reverse is the case at a minimum point.



Tangents drawn at successive points on a curve that has maxima and minima points show that as the abscissa of the moving point increases the tangent turns clockwise through zero angle at a maximum point until an inflexion point is reached when it turns in the opposite direction through zero angle again at a minimum point; and then it may without any inflexion point turn through a right angle at a maximum point, continuing to turn anti-clockwise until possibly at an inflexion point it turns back clockwise through a minimum point when the angle is a right angle again, becoming less as the tangent continues to turn clockwise. The points on a graph at which the ordinate ceases to increase and begins to decrease, or else ceases to decrease and begins to increase are sometimes called *turning points* of the graph, and the corresponding values of the function *turning values*. The turning values are evidently maxima and minima values and the turning points maxima and minima points. While the tangent at an inflexion point turns in opposite directions, the curve is either rising or falling on *both* sides of the point; but at a turning point the curve is rising at a maximum and then falling, or falling at a minimum and then rising. These considerations make it obvious that at a maximum, the angle made by the tangent decreasing, its rate, the second derivative of the function, is negative, and that at a minimum, the angle increasing, the second derivative is positive.



86. Polar Curves. — A polar curve is concave or convex to the pole at a point, according as the tangent to the curve

at the point does not, or does lie on the same side of the curve as the pole. It may be seen from the figure that when a polar curve, as *mn*, is concave to the pole, p or ON increases as ρ increases; hence, the rate of change of p with respect to ρ, $\frac{dp}{d\rho}$ is positive.

When a curve, as *st*, is convex to the pole, p decreases as ρ increases; hence, $\frac{dp}{d\rho}$ is negative.

It follows that *a polar curve is concave or convex to the pole at a point according as* $\frac{dp}{d\rho}$ *is positive or negative.*

At a point of inflexion on a polar curve, as P on mt, $\frac{dp}{d\rho}$ changes sign, and therefore p is a maximum or a minimum; and conversely. Hence to test a polar curve for points of inflexion, p is tested for maxima and minima.

Example. — Examine the Lituus for points of inflexion.

Here $$p = \frac{\rho^2}{\sqrt{\rho^2 + (d\rho/d\theta)^2}} = \frac{2\,a^2\rho}{\sqrt{4\,a^4 + p^4}};$$

$$\therefore \quad \frac{dp}{d\rho} = \frac{2\,a^2\,(4\,a^4 - \rho^4)}{(4\,a^4 + \rho^4)^{\frac{3}{2}}} = 0; \quad \therefore \quad \rho = a\sqrt{2}.$$

Hence, $\rho = a\sqrt{2}$ makes p a maximum; and $\left(a\sqrt{2},\ \frac{1}{2}\right)$ is a point of inflexion.

The spiral $\rho = a^\theta$ has no point of inflexion, since $\frac{dp}{d\rho}$ is always positive.

DETERMINATION OF POINTS OF INFLEXION.

1. Examine $y = x^3 - 3\,x^2 - 9\,x + 9$; for points of inflexion.

$$\frac{dy}{dx} = 3\,x^2 - 6\,x - 9,$$

$$\frac{d^2y}{dx^2} = 6\,x - 6 = 6\,(x - 1) = 0, \quad \therefore \quad x = 1,$$

is abscissa of an inflexion point. The point is (1, −2), to the right of which the curve is concave upward.

2. Examine $x^3 - 3\,bx^2 + a^2y = 0$, for points of inflexion.

Ans. $(b, 2\,b^3/a^2)$ is a point of inflexion, or of maximum slope, to the right of which the curve is concave downward.

3. Examine $y = c \sin x$ for points of inflexion.

Ans. $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\,\pi, 0)$ . . . .

4. Examine the Witch of Agnesi, $y = \dfrac{8\,a^3}{x^2 + 4\,a^2}$, for inflexion points

Ans. $(\pm 2\,a/\sqrt{3}, 3\,a/2)$; concave downward between these points concave upwards outside of them.

Find the points of inflexion of the following curves:

5. $(x/a)^2 + (y/b)^{\frac{2}{3}} = 1$. *Ans.* $x = \pm a/\sqrt{2}$.

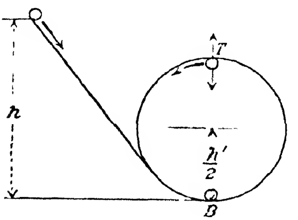
6. $y = (x^2 + x)\,e^{-x}$. *Ans.* $x = 0$ and $x = 3$.

7. $y = e^{-ax} - e^{-bx}$. *Ans* $\dfrac{2\,(\log a - \log b)}{a - b}$.

8. $y = x^3/a^2 + x^2$.

Ans. $(0, 0)$, $(a\sqrt{3}, 3\,a\sqrt{\tfrac{3}{4}})$, $(-a\sqrt{3}, -3\,a\sqrt{\tfrac{3}{4}})$.

87. The Centrifugal Railway. — The centrifugal railway is an example of a simple circular pendulum where the cord of suspension is replaced by a track.



Neglecting the resistance of friction and of the air, the forces acting on the car are the force of gravity and the normal reaction of the track. If h' is the diameter of the circular track and h the height from which the car starts from rest, find the relation between h and h', so that the car will make a complete revolution without leaving the track.

The centrifugal reaction at the highest point of the track must be great enough to overbalance the weight of the car The velocity at B is $v = \sqrt{2\,gh}$; at T, $v = \sqrt{2\,g\,(h - h')}$; whence (Note Art. 70),

$$\frac{Wv^2}{gh'/2} = \frac{W}{gh'/2} \cdot 2\,g\,(h - h') = W, \qquad [(2)\text{ Art. } 184]$$

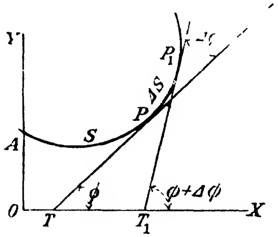
giving $4\,(h - h') = h'$; hence $h = \frac{5}{4}\,h'$, to balance; and, therefore, h should be greater than $\frac{5}{4}\,h'$, for the car to complete the revolution without leaving the track.

CHAPTER VI.

CURVATURE. EVOLUTES.

88. Curvature. — The flexion (Art. 13), $b = \dfrac{dm}{dx} = \dfrac{d^2y}{dx^2}$, being the rate of change of the tangent of the angle made with the x-axis by the tangent to a curve, is one measure of the bending of the curve at the point of tangency. This measure, however, is dependent upon the position of the axes and would change if the axes were rotated.

There is a measure of the bending called the *curvature*, which does not depend upon the choice of the axes, as it is expressed in terms that are the same after the axes are rotated, or even before any axes are drawn. The curvature is denoted by $K = \dfrac{d\phi}{ds}$, the rate of change of the angle of inclination $\phi = \tan^{-1} m$, with respect to the length of arc s.



Thus, let P and P_1 be two points on a plane curve, ϕ and $\phi + \Delta\phi$ the angles which the tangents at P and P_1 make with the x-axis, s the arc AP measured from some fixed point A on the curve up to P, and Δs the arc PP_1. The angle ϕ is in radians, and $\Delta\phi$ is evidently the angle between the two tangents.

The angle $\Delta\phi$ is the *total curvature* of the arc Δs, as it is a measure of the deviation from a straight line of that portion of the curve between the points P and P_1. The sharper the

bending of the curve between the two points the greater is $\Delta\phi$ for equal values of Δs.

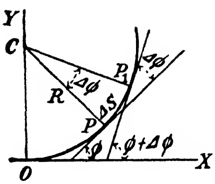
The *average curvature* of the arc Δs is defined as $\frac{\Delta\phi}{\Delta s}$, and is, therefore, the average change per unit length of arc in the inclination of the tangent line.

The limit of $\frac{\Delta\phi}{\Delta s}$, when P_1 approaches P as its limiting position, is called the curvature of the curve at P; that is, the curvature at a point on a curve is $K = \lim\limits_{\Delta s \doteq 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}$.

Otherwise, by rates, the curvature of any curve, as APP_1 at any point, as P, is the s-rate at which the curve bends at P, or the s-rate at which the tangent revolves, where s denotes the length of the variable arc AP. If ϕ denotes (in radians) the variable angle XTP as P moves along the curve APP_1, then, evidently, the curvature of APP_1 at P equals the s-rate of ϕ; that is, $K = \frac{d\phi}{ds}$.

89. Curvature of a Circle. — For a circle of radius R, $\Delta s = R\Delta\phi$ and therefore,

$$\frac{\Delta\phi}{\Delta s} = \frac{1}{R}; \quad \frac{d\phi}{ds} = \frac{1}{R},$$



since the ratio of the increments is constant; that is, the average curvature of any arc of a circle is equal to the curvature at any point of that circle. In other words, a circle is a curve of constant curvature and its curvature is equal to the reciprocal of its radius; that is, the curvature of a circle equals $1/R$ radians to a unit of arc.

For example, if $R = 2$, the circle bends uniformly at the rate of $\frac{1}{2}$ radian to a unit of arc.

If $R = \frac{1}{2}$, the curvature of the circle is 2 radians per unit of arc.

If $R = 1$, for the circle of unit radius the curvature is evidently a radian per unit of arc.

90. Circle, Radius, and Center of Curvature. — The curvature of any curve except the circle varies from one point to another. A circle tangent to a curve and having the same curvature as the curve at the point of contact, therefore, having a radius equal to the reciprocal of the curvature at that point, is called the *circle of curvature* at that point; its radius is called the *radius of curvature;* and its center, the *center of curvature.*

If R denotes the radius of the circle of curvature at any point of a curve, then, since the curvature of the curve is $\frac{d\phi}{ds}$ and equals the curvature of the circle, it follows that,

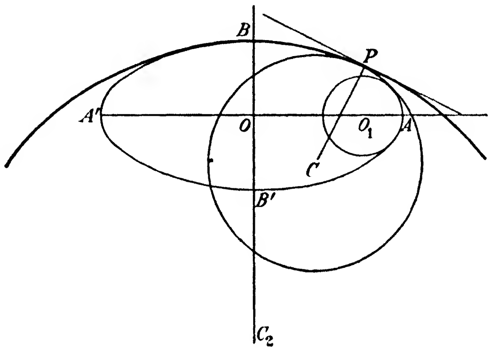
$$\frac{d\phi}{ds} = \frac{1}{R}, \quad \text{and} \quad R = \frac{ds}{d\phi}.$$

If at P (Art. 72, figure) the *direction* of the path of (x, y) became *constant*, (x, y) would trace the tangent at P, and ds might be represented by a length on the tangent; while if at P the *change* of direction of the path became *uniform* with respect to s, (x, y) would trace the circle of curvature at P, and ds would represent an arc of the circle, since it equals $R\,d\phi$, where $d\phi = \Delta\phi$ would be the constant change of angle at center of the circle of curvature.

Thus it is that the curvature is uniform when, as the moving point passes over equal arcs, the tangent turns through equal angles; or conversely; and, as this is the case with the circle only, it is the only curve of uniform curvature.

For any curve the measure of the curvature at a point is the limit of the ratio between the angle described by the tangent and the arc described by the point of contact, as that arc approaches zero; and this limit $\frac{d\phi}{ds}$ equals the

reciprocal of the radius of curvature at the point; hence, the radius of curvature is $\frac{ds}{d\phi}$ and equals the reciprocal of the curvature. The figure shows the circle of curvature for the point $P(x, y)$ of the ellipse; C is the center of curvature, and CP the radius of curvature. It is to be noticed that the



circle of curvature crosses the ellipse at P, and this must be so; for at P the circle and ellipse have the same curvature, but towards A the curvature of the ellipse increases, while that of the circle remains the same, being constant. Hence on the side of P towards the vertex A the circle is outside of the ellipse. From P towards B the curvature of the ellipse decreases, and, therefore, on the side of P towards the vertex B the circle is inside of the ellipse.

So, in general, *the circle of curvature crosses the curve at the point of contact.*

The only exceptions to this rule are at points of maximum and minimum curvature, as the vertices of the ellipse. From A along the curve in either direction, the curvature of the ellipse decreases; hence the circle of curvature at A lies entirely within the ellipse. From B the curvature of the ellipse increases in each direction and so the circle of curvature at B lies entirely without the ellipse.

91. Radius of Curvature in Rectangular Coördinates. — Since $\phi = \tan^{-1} m$, and since $ds^2 = dx^2 + dy^2$;

$$d\phi = \frac{dm}{1 + m^2} \quad \text{and} \quad ds = \sqrt{1 + m^2}\, dx, \quad \text{where} \quad m = \frac{dy}{dx};$$

hence the radius of curvature

$$R = \frac{ds}{d\phi} = \frac{\sqrt{1 + m^2}\, dx}{\dfrac{dm}{1 + m^2}} = \frac{(1 + m^2)^{\frac{3}{2}}}{\dfrac{dm}{dx}}$$

$$= \frac{(1 + m^2)^{\frac{3}{2}}}{b} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}. \tag{1}$$

R will be positive or negative according as $\dfrac{d^2y}{dx^2}$ is positive or negative, if ϕ is always taken as the acute angle which the tangent makes with the x-axis; for then, whether ϕ is positive or negative, $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \sec^3 \phi$ will be positive and the sign of R will be that of $\dfrac{d^2y}{dx^2}$. Hence the sign of R will be plus or minus according as the curve at the point is concave upward or downward. R may be in the form $\dfrac{\sec^3 \phi}{b}$.

If the reciprocals of the members of (1) are taken, then $K = \dfrac{d\phi}{ds} = \dfrac{d^2y}{dx^2} \Big/ \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$, which may be in the form $b \cos^3 \phi$.

If $\dfrac{d^2y}{dx^2}$ is zero at any point of a curve, then $K = \dfrac{1}{R}$ is zero and R is infinite. Thus at a point of inflexion R is infinite. It may be noted that as a curve approaches being a straight line, its curvature approaches zero and its radius of curvature becomes infinite, that is, it increases in length without limit. So a straight line is the line that the arc of a circle of curva-

ture approaches as the radius of the circle increases without limit. On the contrary as the radius of curvature at a point approaches zero, the curvature at the point becomes infinite and the curve will approach a mere point, since the circle of curvature will diminish with zero as a limit for its radius.

92. Approximate Formula for Radius of Curvature. — Since

$$K = \frac{b \times 1}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}},$$

it is seen that the flexion when multiplied by the factor $1/(1 + m^2)^{\frac{3}{2}}$ gives a measure of the bending of a curve independent of the position of the axes.

The flexion is the rate of change of the *tangent* of the inclination of the curve at a point with respect to the *abscissa*, while the curvature is the rate of change of the *inclination* of the curve at a point with respect to the *arc*, where the inclination of the curve is that of the tangent line at the point.

However, when the curve deviates but slightly from a horizontal straight line, the curvature is approximately the same as the flexion, since the slope $m = \frac{dy}{dx}$ being small, $\left(\frac{dy}{dx}\right)^2$ is very small compared with 1, and therefore the formula becomes approximately

$$K = \frac{d^2y}{dx^2}.$$

This approximation for the curvature is used to advantage in the flexure of beams and columns. The approximate formula for the radius of curvature is consequently

$$R = \frac{1}{\frac{d^2y}{dx^2}}.$$

EXERCISE XII.

1. To find R and the curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

Substituting these values in (1), Art. 91, gives

$$R = \left(1 + \frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}\left(-\frac{a^2y^3}{b^4}\right) = -\frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4};$$

$$\therefore \quad K = \frac{d\phi}{ds} = \frac{1}{R} = -\frac{a^4b^4}{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}.$$

The maximum curvature is a/b^2, at A $(a, 0)$, where $R = b^2/a$ is a minimum, and the minimum curvature is b/a^2, at B $(0, b)$, where $R = a^2/b$ is a maximum. (See Art. 90, figure; Art. 97, figure.)

2. To find R and the curvature of the parabola $y^2 = 4\,px$.

$$\frac{dy}{dx} = \frac{2\,p}{y}, \quad \frac{d^2y}{dx^2} = -\frac{4\,p^2}{y^3}.$$

Substituting these values in (1) of Art. 91 gives

$$R = \left(\frac{y^2 + 4\,p^2}{y^2}\right)^{\frac{3}{2}}\left(-\frac{y^3}{4\,p^2}\right) = -\frac{2\,(x + p)^{\frac{3}{2}}}{p^{\frac{1}{2}}};$$

$$\therefore \quad K = \frac{d\phi}{ds} = \frac{1}{R} = -\frac{p^{\frac{1}{2}}}{2\,(x + p)^{\frac{3}{2}}}.$$

At the vertex $(0, 0)$, $R = 2\,p$, the minimum radius; and the maximum curvature is $(1/2\,p)$ radian to a unit of arc. (See Example 1, Art. 97.)

Since $\frac{d^2y}{dx^2} = -\frac{4\,p^2}{y^3}$ is negative for positive values of y and positive for negative values, the curve is concave downward at points whose ordinates are positive, and concave upward at points whose ordinates are negative. The sign of R may be neglected, since the sign of $\frac{d^2y}{dx^2}$ will indicate whether the curve is concave upward or downward at any point.

3. To find R of the cycloid $x = a \operatorname{vers}^{-1}(y/a) \mp \sqrt{2\,ay - y^2}$.

$$\frac{dy}{dx} = \frac{\sqrt{2\,ay - y^2}}{y}, \quad \frac{d^2y}{dx^2} = -\frac{a}{y^2}.$$

Substituting these values in (1) of Art. 75 gives

$$R = \left(1 + \frac{2\,ay - y^2}{y^2}\right)^{\frac{3}{2}}\left(-\frac{y^2}{a}\right) = \left(\frac{2\,a}{y}\right)^{\frac{3}{2}}\left(-\frac{y^2}{a}\right) = -2\sqrt{2\,ay}.$$

At the highest point, $y = 2\,a$, and, therefore, $R = 4\,a$, the maximum of R. At the vertex $(0, 0)$, $R = 0$, and also at other points where y is zero; therefore, R being zero, at those points, which are cusps, the curvature is infinite. (See Example 3, Art. 97, figure.)

Find R and the curvature of each of the following curves.

4. The equilateral hyperbola $2\,xy = a^2$. $\qquad R = (x^2 + y^2)^{\frac{3}{2}}/a^2.$

5. The cubical parabola $y^3 = a^2x$. $\qquad K = \dfrac{d\phi}{ds} = \dfrac{6\,a^4y}{(9\,y^4 + a^4)^{\frac{3}{2}}}.$

6. The logarithmic curve $y = b^x$. $\qquad \dfrac{d\phi}{ds} = \dfrac{my}{(m^2 + y^2)^{\frac{3}{2}}}.$

7. The catenary $y = \dfrac{a}{2}\,(e^{x/a} + e^{-x/a})$. $\qquad \dfrac{d\phi}{ds} = \dfrac{a}{y^2}.$

8. The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. $\qquad R = 3\,(axy)^{\frac{1}{3}}.$

9. The curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. $\qquad K = \dfrac{d\phi}{ds} = \dfrac{a^{\frac{1}{2}}}{2\,(x + y)^{\frac{3}{2}}}.$

93. Radius of Curvature in Polar Coördinates. — From (4) and (10) of Art. 67,

$$\phi = \theta + \psi, \quad \psi = \tan^{-1}\frac{\rho}{d\rho/d\theta};$$

$$\therefore \quad \frac{d\phi}{d\theta} = 1 + \frac{d\psi}{d\theta}, \quad \frac{d\psi}{d\theta} = \frac{(d\rho/d\theta)^2 - \rho \cdot d^2\rho/d\theta^2}{\rho^2 + (d\rho/d\theta)^2};$$

$$\therefore \quad \frac{d\phi}{d\theta} = \frac{\rho^2 + 2\,(d\rho/d\theta)^2 - \rho \cdot d^2\rho/d\theta^2}{\rho^2 + (d\rho/d\theta)^2};$$

$$\therefore \quad R = \frac{ds/d\theta}{d\phi/d\theta} = \frac{[\rho^2 + (d\rho/d\theta)^2]^{\frac{3}{2}}}{\rho^2 + 2\,(d\rho/d\theta)^2 - \rho \cdot d^2\rho/d\theta^2}. \qquad \text{Art. 77 (3)}$$

Corollary. — Since $R = \infty$ at a point of inflexion,

$$\rho^2 + 2\,(d\rho/d\theta)^2 - \rho \cdot \frac{d^2\rho}{d\theta^2} = 0$$

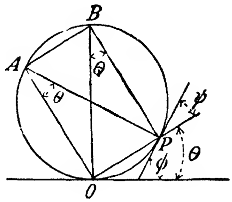
is a necessary condition for such a point.

Example. — To find R for the curve $\rho = \sin\theta$. Here

$$\frac{d\rho}{d\theta} = \cos\theta, \quad \frac{d^2\rho}{d\theta^2} = -\sin\theta;$$

$$\therefore \quad R = \frac{(\rho^2 + \cos^2\theta)^{\frac{3}{2}}}{\rho^2+2\cos^2\theta-\rho\,(-\sin\theta)} = \frac{(\sin^2\theta + \cos^2\theta)^{\frac{3}{2}}}{\sin^2\theta+2\cos^2\theta+\sin^2\theta} = \frac{1}{2}.$$

This curve $\rho = \sin\theta$, a circle with unit diameter, in connection with the formula for polar curves, $\tan\psi = \dfrac{\rho}{d\rho/d\theta}$, furnishes a derivation of the $d\,(\sin\theta)$. Since for circle



$$\psi = \theta, \quad \tan\theta = \frac{\rho}{d\rho/d\theta};$$

$$\therefore \quad d\rho = \frac{\rho\,d\theta}{\tan\theta} = \frac{\sin\theta}{\tan\theta}d\theta = \cos\theta\,d\theta;$$

that is, $\quad d\,(\sin\theta) = \cos\theta\,d\theta.$

Also from figure:

$$\tan\theta = \frac{OP}{OA} = \frac{\rho}{\cos\theta},$$

and since

$$\tan\theta = \frac{\rho}{d\rho/d\theta},$$

$$\therefore \quad \frac{d\rho}{d\theta} = \cos\theta = \text{subnormal } OA;$$

so again, $\quad d\,(\sin\theta) = \cos\theta\,d\theta.$

This curve serves as an illustration of maxima and minima in polar coördinates. Thus, $\rho = \sin\theta$ will be a maximum or a minimum when $\dfrac{d\rho}{d\theta} = \cos\theta = 0$, $\therefore$ when $\theta = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$; and since $\dfrac{d^2\rho}{d\theta^2} = -\sin\theta$, is negative when $\theta = \dfrac{\pi}{2}$, $\rho = \sin\dfrac{\pi}{2} = 1$ is a maximum, while $\dfrac{d^2\rho}{d\theta^2} = -\sin\theta$ is posi-

tive when $\theta = \frac{3\pi}{2}$, $\therefore\ \rho = \sin\frac{3\pi}{2} = -1$ is a minimum

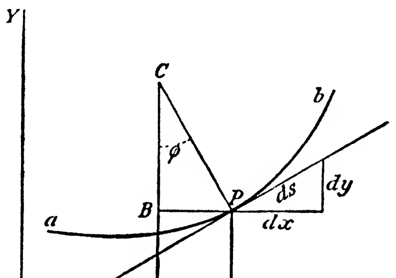
As the denominator of the fractional value of R is 2 for any value of θ, there is no inflexion point, R not being infinite at any point.

EXERCISE XIII.

Find R in each of the following curves:

1. The Cardioid $\rho = a(1 - \cos\theta)$. $\qquad R = 2\sqrt{2a\rho/3}$.
2. The Lemniscate $\rho^2 = a^2\cos 2\theta$. $\qquad R = a^2/3\rho$.
Where is an inflexion point?
3. The Spiral of Archimedes $\rho = a\theta$. $\qquad R = a\frac{(\theta^2+1)^{\frac{3}{2}}}{\theta^2+2}$.
4. The Logarithmic Spiral $\rho = a^\theta$. $\qquad R = \rho\sqrt{1 + (\log a)^2}$.

94. Coördinates of Center of Curvature. — Let $P(x, y)$ be any point on the curve ab, and $C(\alpha, \beta)$ the corresponding



center of curvature. Then PC is R and is perpendicular to the tangent PT. Hence

$$\angle BCP = \angle XTP = \phi,$$

$$OA = OM - BP, \quad AC = MP + BC;$$

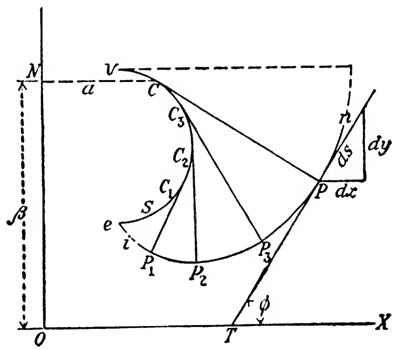
that is, $$\alpha = x - R\sin\phi, \quad \beta = y + R\cos\phi; \tag{1}$$

or $$\alpha = x - R\frac{dy}{ds}, \quad \beta = y + R\frac{dx}{ds}. \tag{2}$$

Substituting in (2), the values of R and ds, gives

$$\alpha = x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}, \quad \beta = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{3}$$

95. Evolutes and Involutes. — Every point of a curve, as *in*, has a corresponding center of curvature. As the point (x, y) moves along the curve *in*, by equation (3) above, the



point (α, β) will trace another curve, as *ev*. The curve *ev*, which is the locus of the centers of curvature of *in*, is called the *evolute* of *in*.

To express the inverse relation, *in* is called an *involute* of *ev*. The figure shows an arc of an involute of a circle.

96. Properties of the Involute and Evolute. — I. Since

$$dx/ds = \cos\phi, \quad dy/ds = \sin\phi, \quad \text{and} \quad ds = R\,d\phi,$$

$$dx = \cos\phi\, ds = R\cos\phi\, d\phi, \tag{1}$$

and

$$dy = \sin\phi\, ds = R\sin\phi\, d\phi. \tag{2}$$

Differentiating equations (1) of Art. 94, and using the relations given in (1) and (2), there results

$$d\alpha = dx - R\cos\phi\, d\phi - \sin\phi\, dR = -\sin\phi\, dR, \tag{3}$$

$$d\beta = dy - R\sin\phi\, d\phi + \cos\phi\, dR = \cos\phi\, dR. \tag{4}$$

Dividing (4) by (3) gives

$$d\beta/d\alpha = -\cot\phi = -dx/dy.$$

That is, *the normal to the involute at* (x, y), *as* P (Art. 95, figure), *is tangent to the evolute at the corresponding point* (α, β), *as* C.

II. Squaring and adding (3) and (4) gives

$$d\alpha^2 + d\beta^2 = dR^2.$$

Let s denote the length of an arc of the evolute; then,

$$d\alpha^2 + d\beta^2 = ds^2.$$

Hence, $$ds = \pm\, dR; \quad \therefore \quad \Delta s = \pm\, \Delta R.$$

That is, the difference between two radii of curvature, as C_3P_3 and C_1P_1 (Art. 95, figure), is equal to the corresponding arc of the evolute, C_1C_3.

These two properties show that the involute *in* can be described by a point in a string unwound from the evolute *ev*. From this property the *evolute* receives its name.

It may be noted that a curve has only one evolute, but an unlimited number of involutes, as each point on the string which is unwound would describe an involute.

97. To Find the Equation of the Evolute of a Given Curve. — Differentiating the equation of the given curve and using equations (3) of Art. 94, α and β will be expressed in terms of x and y. These two equations and that of the given curve furnish three equations between α, β, x and y. Eliminating x and y from these equations, a relation between α and β is obtained, and this relation is the equation of the *evolute* of the given curve, which would itself be *an involute* of the curve found.

Examples. — Find the equation of the evolute of the following curves:

1. The parabola $$y^2 = 4\,px. \tag{1}$$

$$\frac{dy}{dx} = \frac{2\,p}{y}, \quad \frac{d^2y}{dx^2} = -\frac{4\,p^2}{y^3}.$$

Substituting these values in (3) of Art. 94 gives

$$\alpha = 3x + 2p; \quad \beta = -y^3/4p^2;$$

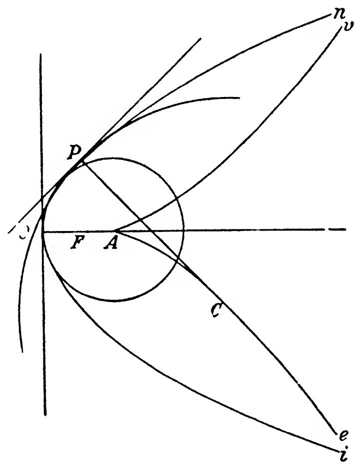
$$\therefore \quad x = (\alpha - 2p)/3, \quad y = -\sqrt[3]{4\beta p^2}.$$

Substituting these values of x and y in (1) gives

$$\beta^2 = 4(\alpha - 2p)^3/27p, \tag{2}$$

as the equation of the evolute of $y^2 = 4px$.

The locus of (2) is the semi-cubical parabola. Thus, if iOn is the locus of (1), F being the focus, then eAv is the locus of (2), where $OA = 2p = 2OF$ is the minimum radius of curvature at O, the point of maximum curvature on the parabola. (Example 2, Exercise XII.)



2. The ellipse

$$a^2y^2 + b^2x^2 = a^2b^2. \tag{1}$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

Substituting these values in (3) of Art. 94 gives

$$\alpha = \frac{(a^2 - b^2)x^3}{a^4}, \quad \beta = -\frac{(a^2 - b^2)y^3}{b^4};$$

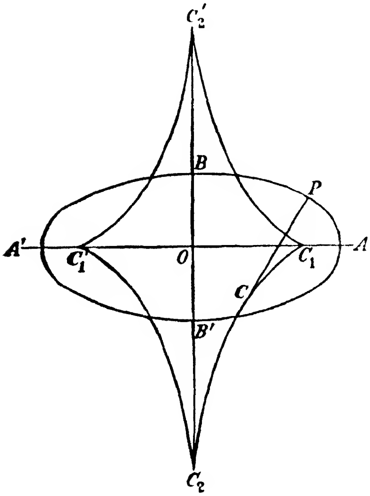
$$\therefore \quad x^2 = \left(\frac{a^4\alpha}{a^2 - b^2}\right)^{\frac{2}{3}}, \quad y^2 = \left(\frac{b^4\beta}{a^2 - b^2}\right)^{\frac{2}{3}}.$$

Hence, the equation of the evolute of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ is

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

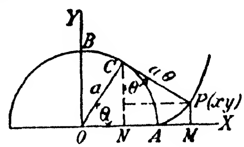
The evolute is $C_1C_2C_1'C_2'$. C_1 is center of curvature for

A; C for P; C_2 for B; C_1' for A'; C_2' for B'. In the figure shown $a = 2b$; when $a = b\sqrt{2}$, then the center of curvature for B is at B' and for B' at B. When $a < b\sqrt{2}$, the centers for B and B' are within the ellipse. The points C_1, C_2, C_1', and C_2' are cusps. The length of the evolute is evidently four times the difference between R at $B(a, b)$, and R at $A(a, o)$; that is, (1, Exercise XII), $4(a^2/b - b^2/a) = 4(a^3 - b^3)/ab$.



Corollary. — For circle, since $a = b$, the evolute is a point, the center of the circle.

The involute of the circle is given by the equations,



$$x = a(\cos\theta + \theta\sin\theta),$$
$$y = a(\sin\theta - \theta\cos\theta).$$

AP is the arc of an involute of the circle.

3. The cycloid $x = a \operatorname{vers}^{-1}(y/a) \mp \sqrt{2ay - y^2}$.

$$\frac{dy}{dx} = \frac{\sqrt{2ay - y^2}}{y}, \quad \frac{d^2y}{dx^2} = -\frac{a}{y^2}.$$

Substituting these values in (3) of Art. 94:

$$y = -\beta, \quad x = \alpha = 2\sqrt{-2a\beta - \beta^2};$$

$$\therefore \quad \alpha = a \operatorname{vers}^{-1}(-\beta/a) \pm \sqrt{-2a\beta - \beta^2}. \qquad (1)$$

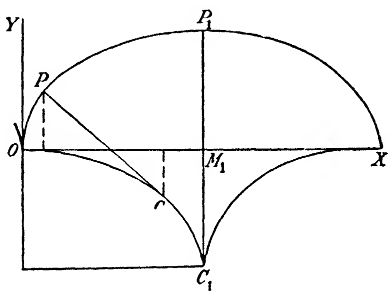
The locus of (1) is another cycloid equal to the given one, the

highest point being at the origin; that is, *the evolute of a cycloid is an equal cycloid.* Thus, the evolute of the arc OP_1 is the arc OC_1, which equals P_1x; and the evolute of P_1x is C_1x, which equals OP_1.

Since $R = 2\sqrt{2\,ay}$ (3, Exercise XII), $C_1P_1 = 4\,a$. Then

$$OP_1X = 2 \cdot OC_1 = 2 \cdot C_1P_1 = 8\,a.$$

Hence, the length of one branch of the cycloid is eight times the radius of the generating circle. (See Example 3, Art. 141.)



If the figure shown be inverted, the principle of the cycloidal pendulum may be perceived. A weight, suspended from C_1 by a flexible cord, may be made to oscillate in the arc OP_1X, by means of some surface shaped like the arcs C_1O and C_1X causing a continuous change in the length of the cord as it comes in contact with the surface. The cycloidal pendulum is isochronal, as the time of an oscillation is independent of the length of the arc. (See Art. 185.)

4. The hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

$$(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

5. The equilateral hyperbola $2\,xy = a^2$.

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = 2\,a^{\frac{2}{3}}.$$

6. Find the length of an arc of the evolute of the parabola $y^2 = 4\,px$ in terms of the abscissas of its extremities.

$$\text{Arc } AC = CP - AO = \frac{2(x+p)^{\frac{3}{2}}}{\sqrt{p}} - 2p \quad \text{(Example 1, figure)}$$

$$= \frac{2}{\sqrt{p}}\left(\frac{\alpha + p}{3}\right)^{\frac{3}{2}} - 2p.$$

7. Show that in the catenary $y = a/2\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$,

$$\alpha = x - y/a\sqrt{y^2 - a^2}, \quad \beta = 2y.$$

8. Find the equation of the evolute of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$(\alpha + \beta)^{\frac{2}{3}} + (\alpha - \beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

CHAPTER VII.

CHANGE OF THE INDEPENDENT VARIABLE. FUNCTIONS OF TWO OR MORE VARIABLES.

98. Different Forms of Successive Derivatives. — As given in Arts. 67, 68, where x is independent dx may be taken as having always the same value and is accordingly treated as a constant; hence,

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}, \quad \frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{d^3y}{dx^3}, \ \ldots.$$

When neither x nor y is independent, $\frac{dy}{dx}$ is a fraction with both numerator and denominator variable, and $d\,dx = d^2x$, etc., hence,

$$\frac{d}{dx}\frac{dy}{dx} = \frac{dx\,d^2y - dy\,d^2x}{dx^3}, \tag{1}$$

$$\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{dx^2\,d^3y - dx\,dy\,d^3x - 3\,dx\,d^2x\,d^2y + 3\,dy\,(d^2x)^2}{dx^5}, \tag{2}$$

. . . .

When y is independent, $d^2y = 0$, $d^3y = 0$, . . . ; hence,

$$\frac{d}{dx}\frac{dy}{dx} = -\frac{dy\,d^3x}{dx^3}, \tag{1$'$}$$

$$\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = \frac{3\,dy\,(d^2x)^2 - dx\,dy\,d^3x}{dx^5}, \tag{2$'$}$$

99. Change of the Independent Variable. — In some applications of the Calculus it is necessary to make a differential equation depend on a new independent variable in place of the one originally selected; that is, there is need *to change the independent variable.*

When $x = \phi(z)$ and it is desired to change the independent variable from x to z; for $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$, respectively, the second members of (1), (2), . . . , above, are substituted; and in the resulting equation, for $x, dx, d^2x, \ldots$, their values gotten from the equation $x = \phi(z)$ are substituted.

Example 1. — Given $y\,d^2y + dy^2 + dx^2 = 0$, in which x is independent, to find the transformed equation in which neither x nor y is independent; also the one in which y is independent.

Dividing both members by dx^2, substituting for $\frac{d^2y}{dx^2}$ the second member of (1) Art. 98, and multiplying both members by dx^3, gives

$$y\,(d^2y\,dx - d^2x\,dy) + (dy^2\,dx + dx^3) = 0,$$

in which neither x nor y is independent.

Putting $d^2y = 0$, and dividing by $-dy^3$, gives

$$y\frac{d^2x}{dy^2} - \frac{dx^3}{dy^3} - \frac{dx}{dy} = 0,$$

in which the position of dy indicates that y is independent.

Example 2. — To change the independent variable from x to θ in

$$R = \frac{[1 + (dy/dx)^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}};$$

given $x = \rho\cos\theta$, $y = \rho\sin\theta$, ρ being a function of θ.

From the data,

$$\begin{aligned} dy &= \sin\theta\,d\rho + \rho\cos\theta\,d\theta, \\ dx &= \cos\theta\,d\rho - \rho\sin\theta\,d\theta, \\ d^2y &= \sin\theta\,d^2\rho + 2\cos\theta\,d\theta\,d\rho - \rho\sin\theta\,d\theta^2, \end{aligned}$$

and

$$d^2x = \cos\theta\,d^2\rho - 2\sin\theta\,d\theta\,d\rho - \rho\cos\theta\,d\theta^2.$$

Substituting these values in value of R and simplifying, gives

$$R = \frac{[\rho^2 + (d\rho/d\theta)^2]^{\frac{3}{2}}}{\rho^2 + 2\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}, \text{ the value of } R \text{ in Art. 93.}$$

To change the independent variable from x to y; for d^2y/dx^2, d^3y/dx^3, . . . , respectively, the second members of (1′), (2′), . . . , above, are substituted; or in the general result, as in example 1, make $d^2y = 0$, $d^3y = 0$, etc.

Example 3. — Change the independent variable from x to y in

$$3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

Substituting for $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$, respectively, the second members of (1′) and (2′) gives after reduction

$$\frac{d^3x}{dy^3} + \frac{d^2x}{dy^2} = 0,$$

in which the position of dy shows that y is independent.

EXERCISE XIV.

1. Given $x = \cos\theta$, change the independent variable from x to θ in

$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0.$ *Ans.* $\frac{d^2y}{d\theta^2} + y = 0.$

2. Given $x = \frac{1}{z}$, change the independent variable from x to z in

$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \frac{a^2}{x^2}y = 0.$ *Ans.* $\frac{d^2y}{dz^2} + a^2y = 0.$

3. Given $x^2 = 4z$, change the independent variable from x to z in

$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0.$ *Ans.* $z\frac{d^2y}{dz^2} + \frac{dy}{dz} + y = 0$

4. Given $x = \cos z$, change the independent variable from x to z in

$(1 - x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx}.$ *Ans.* $\frac{d^2y}{dz^2} = 0.$

5. Change the independent variable from x to y in

$$\frac{d^2y}{dx^2} + (e^y - x)\left(\frac{dy}{dx}\right)^3 = 0. \qquad \textit{Ans.}\ \frac{d^2x}{dy^2} + x = e^y.$$

6. Given $z = \dfrac{x\,dy - y\,dx}{y\,dy + x\,dx}$, to find the transformed equation when $x = \rho\cos\theta$, $y = \rho\sin\theta$, and ρ is independent. *Ans.* $z = \rho\dfrac{d\theta}{d\rho}$.

100. Function of Several Variables. — A function may depend upon two or more variables having no mutual relation, that is, independent of each other. Thus the volume of a gas depends upon the temperature and also upon the pressure to which it is subjected, and the temperature and the pressure may vary independently.

A variable z is a function of the independent variables x, y, . . . when for each set of values of these variables there is determined a definite value or values of z.

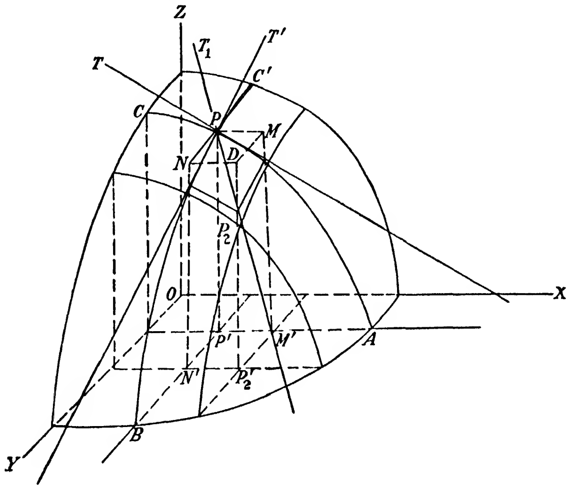
A function of two variables

$$z = f(x, y), \text{ where } x \text{ and } y \text{ are independent,}$$

is represented geometrically by a surface, plane or curved according to the form of the functional relation; and to each pair of values of (x, y) there corresponds a point on this surface. When x and y vary, the point takes another position, and it will take the new position either by x and y varying simultaneously or by one remaining constant while the other changes.

101. Partial Differentials. — A *partial differential* of a function of two or more variables is the differential when only one of the variables is supposed to change. Let $z = f(x, y)$ be the surface shown in the figure, and $P(x, y, z)$ the moving point; then if y is constant while x changes, P will move on the plane curve PA and dx may be represented by PM or $P'M'$; on the other hand, if x is constant while y changes, P will move on the plane curve PB and dy may be represented by PN or $P'N'$. The differential of z as a

function of x, y being regarded as a constant, is denoted by $\partial_x z$; and the differential of z when y alone is variable is denoted by $\partial_y z$. These differentials are the partial differentials of z with respect to x and y, respectively.



Note. — The partial $\partial_x z$ may in the figure be represented by the distance on the ordinate from the point M to the tangent TP, and so too, the partial $\partial_y z$ may be represented by the distance from N to the tangent $T'P$, both being negative in this case as z is decreasing. The $\Delta_x z$ and the $\Delta_y z$ are the distances from the points M and N to the surface curves through P_2.

102. Partial Derivatives. — *The partial derivatives* of z with respect to x and y are denoted by $\dfrac{\partial z}{dx}$ and $\dfrac{\partial z}{dy}$, respectively, and they may be represented by the equivalent notation, $f_x'(x, y)$ and $f_y'(x, y)$.

In the figure of Art. 101, consider P as the intersection of the curves CPA and $C'PB$, cut from the surface by the planes $y = b$ and $x = a$, respectively; then the slope of the

curve CPA is given by the partial derivative $\frac{\partial z}{dx}$, and that of the curve $C'PB$ by the partial derivative $\frac{\partial z}{dy}$; that is, the partial derivatives are the tangents of the inclination of the tangent lines at P to the axes of X and Y, respectively. The values of the slopes for some definite point P on the surface are gotten by substituting in the expressions for $\frac{\partial z}{dx}$ and $\frac{\partial z}{dy}$, respectively, the corresponding values of x and y. Thus in this case (a, b), or P', being the projection of P on the xy-plane, a is substituted for x and b for y.

103. Tangent Plane. Angles with Coördinate Planes. — In the figure of Art. 101, let P be the point (x_1, y_1, z_1); PT, the tangent to CPA in the plane $y = y_1$; and PT', the tangent to $C'PB$ in the plane $x = x_1$.

The equations of PT are

$$z - z_1 = \left[\frac{\partial z}{dx}\right]_1 (x - x_1), \quad y = y_1, \tag{1}$$

and of PT',

$$z - z_1 = \left[\frac{\partial z}{dy}\right]_1 (y - y_1), \quad x = x_1. \tag{2}$$

The plane tangent to the surface at P has for its equation,

$$z - z_1 = \left[\frac{\partial z}{dx}\right]_1 (x - x_1) + \left[\frac{\partial z}{dy}\right]_1 (y - y_1), \tag{3}$$

since it is determined by the two intersecting tangents, is of the first degree with respect to x, y, z, and is satisfied by (1) and (2).

The *equations of the normal* through P are those of a line through (x_1, y_1, z_1) perpendicular to (3). Its equations are

$$x - x_1 \Big/ \left[\frac{\partial z}{dx}\right]_1 = y - y_1 \Big/ \left[\frac{\partial z}{dy}\right]_1 = -(z - z_1). \tag{4}$$

The angles made by the tangent plane with the coordinate planes are equal to the inclinations of the normal to the axes.

The direction cosines of the line perpendicular to (3) are proportional to $\left[\frac{\partial z}{dx}\right]_1, \left[\frac{\partial z}{dy}\right]_1, -1.$

Hence, if α, β, γ, are the inclinations of the normal to OX, OY, OZ, respectively,

$$\cos\alpha \Big/ \left[\frac{\partial z}{dx}\right]_1 = \cos\beta \Big/ \left[\frac{\partial z}{dy}\right]_1 = \cos\gamma / -1. \tag{5}$$

Also
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{6}$$

From (5) and (6), in general, at any point (x, y, z),

$$\cos^2\alpha = \left(\frac{\partial z}{dx}\right)^2 \Big/ 1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2,$$

$$\cos^2\beta = \left(\frac{\partial z}{dy}\right)^2 \Big/ 1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2,$$

$$\cos^2\gamma = 1 \Big/ 1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2. \tag{7}$$

For the inclination of the tangent plane to XY, from (7),

$$\sec^2\gamma = 1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2, \tag{8}$$

and
$$\tan^2\gamma = \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2. \tag{9}$$

From (9), calling the tangent of the angle made by the tangent plane with the plane XY the *slope*,

$$\text{the slope} = \sqrt{\left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2}.$$

Example. — Find the equations of the tangent plane and normal, to the sphere $x^2 + y^2 + z^2 = a^2$, at (x_1, y_1, z_1).

$$\frac{\partial z}{dx} = -\frac{x}{z}, \qquad \frac{\partial z}{dy} = -\frac{y}{z};$$

$$\therefore \quad \left[\frac{\partial z}{dx}\right]_1 = -\frac{x_1}{z_1}, \quad \left[\frac{\partial z}{dy}\right]_1 = -\frac{y_1}{z_1}.$$

Substituting in (3),

$$z - z_1 = -\frac{x_1}{z_1}(x - x_1) - \frac{y_1}{z_1}(y - y_1),$$

$$xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2 = a^2. \quad \textit{Ans.}$$

From (4) for the normal:

$$(x - x_1)\frac{z_1}{x_1} = (y - y_1)\frac{z_1}{y_1} = z - z_1,$$

$$\frac{x}{x_1} - 1 = \frac{y}{y_1} - 1 = \frac{z}{z_1} - 1, \quad \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}. \quad \textit{Ans.}$$

EXERCISE XV.

1. Find the equation of the tangent plane and its *slope*, for the ellipsoid, $x^2 + 2y^2 + 3z^2 = 20$, at $(3, 2, +z_1)$.

Ans. $3x + 4y + 3z = 20$; $\frac{5}{3}$.

2. Find the equation of the tangent plane to the elliptic paraboloid, $z = 3x^2 + 2y^2$, at the point $(1, 2, 11)$.

Ans. $6x + 8y - z = 11$.

3. Find the equations of the tangent plane and normal to the cone, $3x^2 - y^2 + 2z^2 = 0$, at (x_1, y_1, z_1).

Ans. $3xx_1 - yy_1 + 2zz_1 = 0$; $\dfrac{x - x_1}{3x_1} = \dfrac{y - y_1}{-y_1} = \dfrac{z - z_1}{2z_1}$.

Note. — The equations of the tangent plane and normal are illusory if formed for the origin. Every tangent plane to the cone goes through the origin and there is no definite normal at the origin. When at special points on a surface the three partial derivatives of the function with respect to each of the three variables are all zero, there is no definite tangent plane or normal at the point. Such points are called *conical* points, the vertex of a cone being the typical case.

104. Total Differentials. — When $z = f(x, y)$ is differentiated, both x and y varying, the *total* differential dz or $df(x, y)$ is gotten.

The derivations of the formulas for differentiation of algebraic, logarithmic and exponential functions, given in

Chapter II, hold when u, v, y, and z denote functions of two or more independent variables; hence the total differential of $f(x, y)$ may be gotten by the principles established in those derivations. *The total differential of a function of two or more variables is equal to the sum of its partial differentials.*

If $z = f(x, y)$, then

$$dz = \partial_x z + \partial_y z = \frac{\partial z}{dx} dx + \frac{\partial z}{dy} dy;$$

and if $v = f(x, y, z)$, then,

$$dv = \partial_x v + \partial_y v + \partial_z v = \frac{\partial v}{dx} dx + \frac{\partial v}{dy} dy + \frac{\partial v}{dz} dz,$$

where the last form of the partial differentials is another convenient notation. In the figure of Art. 101, dz is represented by the distance on the ordinate from D to the tangent plane at P and is there negative, Δz being DP_2, which is negative.

The truth of this theorem has been illustrated geometrically in the derivations of $d(uy)$ and $d(xyz)$ in Arts. 28 and 29, and the theorem is readily established analytically. Thus, it has been found that all the terms of $d(f(x, y))$ are of the first degree in dx and dy; hence, if $z = f(x, y)$,

$$dz = \phi(x, y)\, dx + \phi_1(x, y)\, dy, \tag{1}$$

where $\phi(x, y)$ and $\phi_1(x, y)$ denote, respectively, the sums of the coefficients of dx and dy in the several terms of dz. When x alone varies, (1) becomes

$$\partial_x z = \phi(x, y)\, dx. \tag{2}$$

When y alone varies, (1) becomes

$$\partial_y z = \phi_1(x, y)\, dy. \tag{3}$$

Hence, from (1), (2) and (3),

$$dz = \partial_x z + \partial_y z = \frac{\partial z}{dx} dx + \frac{\partial z}{dy} dy.$$

105. If $z = f(x, y) = c$, $$\frac{dy}{dx} = -\frac{\partial z/dx}{\partial z/dy}, \tag{1}$$

for $$\frac{\partial z}{dx}dx + \frac{\partial z}{dy}dy = dz = d(c) = 0, \tag{2}$$

which solved for $\frac{dy}{dx}$ gives (1).

This formula for the derivative of an implicit function is useful in many cases.

Example. — Given $x^4 - a^2xy + b^2y^2 = c = z$, to find dy/dx.

Here $$\frac{\partial z}{dx} = 4x^3 - a^2y \quad \text{and} \quad \frac{\partial z}{dy} = -a^2x + 2b^2y;$$

$$\therefore \quad \frac{dy}{dx} = \frac{4x^3 - a^2y}{a^2x - 2b^2y}. \qquad \textit{Ans.} \text{ by (1).}$$

EXERCISE XVI.

1. $u = x^2/a^2 + y^2/b^2$.

$$\frac{\partial u}{dx} = \frac{2x}{a^2}, \qquad \frac{\partial u}{dy} = \frac{2y}{b^2}.$$

$$\therefore \quad \frac{x}{2}\frac{\partial u}{dx} + \frac{y}{2}\frac{\partial u}{dy} = \frac{x^2}{a^2} + \frac{y^2}{b^2} = u.$$

2. $u = b \times y^2 + cx^2 + ey^3$.

$$\frac{\partial u}{dx} = by^2 + 2cx, \qquad \frac{\partial u}{dy} = 2bxy + 3ey^2.$$

3. $u = \dfrac{xy}{x + y}$. $\qquad x \cdot \dfrac{\partial u}{dx} + y \cdot \dfrac{\partial u}{dy} = u.$

4. $u = y^x$. $\qquad \dfrac{\partial u}{dx} + \left(\dfrac{y}{x}\right) \cdot \dfrac{\partial u}{dy} = u(\log y + 1).$

5. $u = \log(e^x + e^y)$. $\qquad \dfrac{\partial u}{dx} + \dfrac{\partial u}{dy} = 1.$

6. $u = bxy^2 + cx^2 + ey^3$. $\qquad du = (by^2 + 2cx)\,dx + (2bxy + 3ey^2)\,dy.$

7. $u = y^x$. $\qquad du = y^x \log y\,dx + xy^{x-1}\,dy.$

8. $u = y^{\sin x}$. $\qquad du = y^{\sin x} \log y \cos x\,dx + y^{\sin x - 1} \sin x\,dy.$

By Art. 105, find dy/dx when:

9. $x^3 + y^3 - 3axy = 0$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2 - ay}{ax - y^2}.$

10. $x^y - y^x = 0$. $\qquad \dfrac{dy}{dx} = \dfrac{y^2 - xy\log y}{x^2 - xy\log x}.$

11. $x \log y - y \log x = 0.$ $\qquad \dfrac{dy}{dx} = \dfrac{y}{x} \cdot \dfrac{x \log y - y}{y \log x - x}.$

12. $y^3 - 2x^2y + bx = a = u.$ $\qquad \dfrac{dy}{dx} = \dfrac{4xy - b}{3y^2 - 2x^2}.$

106. Total Derivatives. — If $u = f(x, y, z)$, $y = \phi(x)$, and $z = \phi_1(x)$, u is *directly* a function of x and indirectly a function of x through y and z. The total differential,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \quad \text{(by Art. 104)}$$

becomes by dividing by dx,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}, \tag{1}$$

where $\dfrac{du}{dx}$ is the total derivative of u as a function of x.

Corollary 1. — If $u = f(y, z)$, $y = \phi(x)$, and $z = \phi_1(x)$,

$$\frac{du}{dx} = \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}. \tag{2}$$

Corollary 2. — If $u = f(y)$ and $y = \phi(x)$,

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}, \tag{3}$$

where $\dfrac{du}{dx}$ is the derivative of a function of a function, and (3) is the formula that is the subject of *Remarks* in Art. 19.

Corollary 3. — If $u = f(x, y, z)$ and x, y and z are independent of each other, they may be regarded as functions of time t; hence, the expression for the total differential du above becomes by dividing by dt,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}, \tag{4}$$

where $\dfrac{du}{dt}$ is the total time-derivative or rate of change of u.

Similarly, when $z = f(x, y)$, x and y being functions of t,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{5}$$

If y is a function of x, as $y = \phi(x)$, putting x for t in (5), gives

$$\frac{dz}{dx} = \frac{\partial z}{dx} + \frac{\partial z}{dy}\frac{dy}{dx}. \tag{6}$$

107. Illustrative Examples. — *Example* 1. — The edges of a right parallelopiped are 6, 8 and 10 feet. They are increasing at the rate of 0.02 foot per second, 0.03 foot per second and 0.04 foot per second, respectively. Show at what rate the volume is increasing.

Let volume $= u = xyz$, then by (4), Art. 106:

$$\begin{aligned}\frac{du}{dt} &= yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt}\\ &= 80 \times 0.02 + 60 \times 0.03 + 48 \times 0.04\\ &= 1.60 + 1.80 + 1.92 = 5.32 \text{ cubic feet per second.}\end{aligned}$$

See Art. 29 where $du\,(= dV)$ is shown geometrically by figure.

Example 2. — Given the formula for gas, $pV = KT$, where p is pressure, V is volume, T is temperature, and K is a constant. Let $K = 50$, and let the volume and temperature at a given time be $V_0 = 5$ cu. ft. and $T_0 = 250°$. The corresponding pressure is

$$p_0 = \frac{50 \times 250}{5} = 2500 \text{ lb. per sq. ft.}$$

If in this state the temperature is rising at the rate of 0.5 degree per minute and the volume is increasing at the rate of 0.2 cu. ft. per minute, required the rate at which the pressure is changing. Here $p = 50\,\dfrac{T}{V}$,

whence
$$\frac{\partial p}{dT} = \frac{50}{V}, \quad \frac{\partial p}{dV} = -50\frac{T}{V^2}.$$

Hence, in the given state,

$$\left.\frac{\partial p}{dT}\right]_{T=T_0} = \frac{50}{5} = 10,$$

and $$\left.\frac{\partial p}{dV}\right]_{V=V_0} = -\frac{50 \times 250}{5^2} = -500.$$

Given $$\frac{dT}{dt} = 0.5 \quad \text{and} \quad \frac{dV}{dt} = 0.2.$$

Then

$$\frac{dp}{dt} = \frac{\partial p}{dT}\frac{dT}{dt} + \frac{\partial p}{dV}\frac{dV}{dt} = 10 \times 0.5 - 500 \times 0.2 = -95,$$

by (5) Art. 106; that is, the pressure is decreasing at the rate of 95 lb. per sq. ft. per min.

Example 3. — A point on an ellipsoid $\frac{x^2}{36} + \frac{y^2}{25} + \frac{z^2}{49} = 1$, in the position $x = 3$, $y = -4$, moves so that x increases at the rate of two units per second, while y decreases at the rate of three units per second. Find the rate of change of z. Here

$$\frac{\partial z}{dx} = -\frac{7x}{36\sqrt{1 - \frac{x^2}{36} - \frac{y^2}{25}}}, \qquad \frac{\partial z}{dy} = -\frac{7y}{25\sqrt{1 - \frac{x^2}{36} - \frac{y^2}{25}}},$$

$$\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = 3;$$

$$\frac{dz}{dt} = -\left.\frac{14x}{36\sqrt{1 - \frac{x^2}{36} - \frac{y^2}{25}}} + \frac{21y}{25\sqrt{1 - \frac{x^2}{36} - \frac{y^2}{25}}}\right]_{3,-4} \quad \text{(by (5) Art. 106)}$$

$$\frac{dz}{dt} = -\frac{679}{15\sqrt{11}} \text{ units per sec., the rate of change of } z.$$

EXERCISE XVII.

1. $u = z^2 + y^3 + zy, \quad z = \sin x, \quad y = e^x;$ find $\frac{du}{dx}$.

Ans. $\frac{du}{dx} = 3e^{3x} + e^x(\sin x + \cos x) + \sin 2x.$

2. $u = \sqrt{x^2 + y^2}, \quad y = mx + c.$ $\qquad \frac{du}{dx} = \frac{(1 + m^2)x + mc}{\sqrt{x^2 + (mx + c)^2}}.$

3. $u = \sin^{-1}(y - z), \quad y = 3x, \quad z = 4x^3.$ $\qquad \dfrac{du}{dx} = \dfrac{3}{\sqrt{1 - x^2}}.$

4. $u = \tan^{-1}\dfrac{y}{x}, \quad x^2 + y^2 = r^2.$ $\qquad \dfrac{du}{dx} = -\dfrac{1}{\sqrt{r^2 - x^2}}.$

5. $u = \log(x + y), \quad y = \sqrt{x^2 + a^2}.$ $\qquad \dfrac{du}{dx} = \dfrac{1}{\sqrt{x^2 + a^2}}.$

6. With the same data as in illustrative Ex. 2, when the pressure of the gas is increasing at the rate of 40 lb. per sq. ft. per sec. and the temperature is falling at the rate of 1 degree per sec., find the rate of change of the volume.

Ans. $\dfrac{dV}{dt} = -0.1$ cu. ft. per sec.

7. A point on a elliptic paraboloid $z = 2x^2 + 5y^2$, in the position $x = -3$, $y = 1$, moves so that the rate of change of x is 3 units per sec., and that of y is 2 units per sec. Find the rate of change of z.

Ans. $\dfrac{dz}{dt} = -16$ units per sec.

108. Approximate Relative Rates and Errors. — The method of Art. 41 for finding the errors or small differences in a function, due to slight variations or inaccuracies in the independent variable, is applicable to a function of two or more variables. Since when an area $A = f(x, y)$, the relative rate of increase of A is

$$\frac{\dfrac{\partial A}{dx}}{A} + \frac{\dfrac{\partial A}{dy}}{A} = \frac{f_x'(x, y)}{A} + \frac{f_y'(x, y)}{A} = \frac{f_{x,y}'(A)}{A};$$

hence,
$$\Delta A = \frac{\partial A}{dx}\Delta x + \frac{\partial A}{dy}\Delta y \tag{1}$$

and
$$\frac{\Delta A}{A} = \frac{\partial A}{dx}\frac{\Delta x}{A} + \frac{\partial A}{dy}\frac{\Delta y}{A} \tag{2}$$

are approximate relations. When, for example, the area of a rectangle is given by $A = xy$, and therefore, $dA = x\,dy + y\,dx$, when x and y are the measurements and dx and dy the errors or inaccuracies, then dA gives the approximate error in area due to the errors dx and dy.

If a rectangle is laid out 1000 ft. on one side and 100 ft. on the other, and the tape is 0.01 ft. too long; then by (1)

$$\begin{aligned}\Delta A &= y \cdot \Delta x + x \cdot \Delta y = 100 \times 0.1 + 1000 \times 0.01 \\ &= 10.00 + 10.00 = 20 \text{ sq. ft.}\end{aligned}$$

is the approximate error and the exact error is 20.001 sq. ft., found by more laborious computation. The approximate relative error is by (2)

$$\frac{\Delta A}{A} = \frac{20}{100{,}000} = \frac{1}{5000},$$

making the percentage error

$$\frac{100\,\Delta A}{A} = \frac{1}{50} \quad \text{or} \quad 0.02 \text{ of } 1 \text{ per cent.}$$

EXERCISE XVIII.

1. In the illustrative Example 1 of Art. 107, suppose the error in measuring the edges was 0.02 ft., 0.03 ft., and 0.04 ft., respectively, find the approximate error in the volume computed with 6, 8 and 10 ft. as the edges. *Ans.* 5.32 cu. ft.

(Exact error, $\Delta V = 5.339624$ cu. ft.)

2. The total surface of a cylinder with diameter equal to altitude is to be gilded at a cost of 10 cents per square inch. If the altitude is measured as 24 in., find the maximum error in cost, measurement being accurate to $\frac{1}{32}$ in.

3. The period of a pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$. Find the greatest error in the period if there is an error of $\pm\frac{1}{10}$ ft. in measuring a 10 ft. L, and g, taken as 32 ft./sec², may be in error $\frac{1}{20}$ ft. per sec². Find the percentage error.

Ans. 0.0204 sec., $\frac{37}{64}$ per cent.

4. In estimating the number of bricks in a pile, if the pile is measured to be $8 \times 50 \times 5$ ft., and the count is 12 bricks to the cubic foot, find the cost of the error when the tape is stretched 2 per cent beyond the standard length, bricks being sold at $10 per thousand.

5. If the side c of a triangle ABC is determined by measuring the sides a and b and the included angle C, show that the error Δc, due to inaccurate measurements, is given approximately by the equation,

$$\Delta c = \Delta a \cos B + \Delta b \cos A + a\Delta C \sin B.$$

6. If the horse power of a steamship is given by the formula $H = Kv^3 D^{\frac{2}{3}}$, show that the increase in horse power, due to an increase Δv in the speed and an increase ΔD in the displacement, is given approximately by the equation,

$$\Delta H = 3\,Kv^2 D^{\frac{2}{3}} \cdot \Delta v + \tfrac{2}{3}\,Kv^3 D^{-\frac{1}{3}} \cdot \Delta D.$$

7. Show that the relative error in the area of the ellipse due to inaccurate measurements of the semi-axes a and b is given approximately by $\dfrac{\Delta A}{A} = \dfrac{b \cdot \Delta a + a \cdot \Delta b}{ab}$.

8. The equation for the length L and the period T of a pendulum being $4\,\pi^2 L = T^2 g$, if L is calculated taking $T = 1$ and $g = 32$ ft./sec², while the true values are $T = 1.02$ and $g = 32.01$ ft./sec², show that the approximate error in L is $\Delta L = 0.0326\ .\ .$ ft., and the percentage error about 4 per cent.

9. In determining specific gravity by the formula $s = A/A - W$, where A is the weight in air and W the weight, find (*a*) approximately the maximum error in s if A can be read within 0.01 lb. and W to 0.02 lb., the actual readings being $A = 9$ lb., $W = 5$ lb., find (*b*) the maximum relative error.

Ans. (*a*) $\Delta s = 0.0144$;

(*b*) $\dfrac{\Delta s}{s} = \dfrac{23}{3600} = \dfrac{23}{36}$ per cent.

109. Partial Differentials and Derivatives of Higher Orders. — If only one of the independent variables is supposed to vary at the same time, by successive differentiations there are formed the *successive partial differentials* $\partial_x{}^2 u,\ \partial_y{}^2 u,\ \partial_x{}^3 u,\ \partial_y{}^3 u,\ .\ .\ .$ or

$$\frac{\partial^2 u}{dx^2}\,dx^2, \quad \frac{\partial^2 u}{dy^2}\,dy^2, \quad \frac{\partial^3 u}{dx^3}\,dx^3, \quad \frac{\partial^3 u}{dy^3}\,dy^3, \ .\ .\ .\ .$$

For example, if
$$u = x^2 + xy^2 + y^2, \tag{1}$$

$$\partial_x u = (2\,x + y^2)\,dx, \qquad \partial_x{}^2 u = 2\,dx^2, \qquad \partial_x{}^3 u = 0;$$
$$\partial_y u = (2\,xy + 2\,y)\,dy, \quad \partial_y{}^2 u = (2\,x + 2)\,dy^2, \quad \partial_y{}^3 u = 0.$$

If u is differentiated with respect to x, then the result with respect to y, there is gotten the *second partial differential,*

$$\partial_{xy}{}^2 u \quad \text{or} \quad \frac{\partial^2 u}{dx\,dy}\,dx\,dy.$$

For example, if $u = x^3 + x^2y^2$, (2)

$$\partial_x u = (3\,x^2 + 2\,xy^2)\,dx, \quad \partial_{xy}{}^2 u = 4\,xy\,dx\,dy.$$

Similarly, the *third* partial differential $\partial_{yx^2}{}^3 u$ or $\frac{\partial^3 u}{dy\,dx^2}\,dy\,dx^2$ denotes the result gotten by differentiating u once with respect to y, then this result twice successively with respect to x.

The symbols for the *partial derivatives* are:

$$\frac{\partial^2 u}{dx^2}, \quad \frac{\partial^2 u}{dx\,dy}, \quad \frac{\partial^2 u}{dy^2}, \quad \frac{\partial^3 u}{dx^3}, \quad \frac{\partial^3 u}{dy\,dx^2}, \quad \ldots .$$

In getting the successive partial differentials and derivatives of u or $f(x, y)$, dx and dy are treated as constants, since x and y are independent variables, varying by uniform increments. The equivalent symbols for the higher partial derivatives by another notation are for $f(x, y)$,

$$f_x{}''(x, y),\ f_{xy}{}''(x, y),\ f_y{}''(x, y),\ f_x{}'''(x, y),\ f_{yx^2}{}'''(x, y),\ \ldots .$$

110. Interchange of Order of Differentiation. —

If $$u = f(x, y), \quad \frac{\partial^2 u}{dx\,dy} \equiv \frac{\partial^2 u}{dy\,dx}, \tag{1}$$

$$\frac{\partial^3 u}{dx^2\,dy} \equiv \frac{\partial^3 u}{dx\,dy\,dx} \equiv \frac{\partial^3 u}{dy\,dx^2}, \text{ etc.}; \tag{2}$$

that is, *if u is differentiated successively m times with respect to x and n times with respect to y, the result is independent of the order of these differentiations.*

It can be shown that the order is always a matter of indifference if $f_x{}'(x, y)$, $f_{xy}{}''(x, y)$ or $f_y{}'(x, y)$, $f_{yx}{}''(x, y)$ are continuous functions of the two variables (x, y) taken together.

In most cases that call for the application of the methods of the Calculus to physical problems the partial derivatives give the same result in whatever order the differention is done.

For example, to verify the theorem in some cases:

Example 1. — Given $u = e^x \cos y$;

$$\frac{\partial u}{dx} = e^x \cos y, \qquad \frac{\partial}{dy}\left(\frac{\partial u}{dx}\right) = \frac{\partial^2 u}{dy\,dx} = -e^x \sin y;$$

$$\frac{\partial u}{dy} = -e^x \sin y, \qquad \frac{\partial}{dx}\left(\frac{\partial u}{dy}\right) = \frac{\partial^2 u}{dx\,dy} = -e^x \sin y.$$

Example 2. — Given $u = \dfrac{x \log z}{y}$;

$$\frac{\partial u}{dx} = \frac{\log z}{y}, \qquad \frac{\partial^2 u}{dz\,dx} = \frac{1}{yz};$$

$$\frac{\partial u}{dz} = \frac{x}{yz}, \qquad \frac{\partial^2 u}{dx\,dz} = \frac{1}{yz};$$

$$\frac{\partial u}{dy} = -\frac{x \log z}{y^2}, \qquad \frac{\partial^2 u}{dx\,dy} = -\frac{\log z}{y^2};$$

$$\frac{\partial u}{dx} = \frac{\log z}{y}, \qquad \frac{\partial^2 u}{dy\,dx} = -\frac{\log z}{y^2};$$

$$\frac{\partial u}{dz} = \frac{x}{yz}, \qquad \frac{\partial^2 u}{dy\,dz} = -\frac{x}{y^2 z};$$

$$\frac{\partial u}{dy} = -\frac{x \log z}{y^2}, \qquad \frac{\partial^2 u}{dz\,dy} = -\frac{x}{y^2 z}.$$

EXERCISE XIX.

Verify the identities (1) and (2) of Art. 110 in each of the following nine examples:

1. $u = \cos(x + y)$. **2.** $u = e^x \sin y$. **3.** $u = \cos xy^2$.

4. $u = x^3y^2 + ay^3$. **5.** $u = \log(x^2 + y^2)$. **6.** $u = y^x$.

7. $u = xy\cos(x + y)$. **8.** $u = \tan^{-1}\dfrac{x}{y}$. **9.** $u = \sin^2 x \cos y$.

10. If $u = (x + y)^2$, $\qquad x\dfrac{\partial^2 u}{dx^2} + y\dfrac{\partial^2 u}{dx\,dy} = \dfrac{\partial u}{dx}$.

11. If $u = (x^2 + y^2)^{\frac{1}{3}}$, $\qquad x^2\dfrac{\partial^2 u}{dx^2} + 2\,xy\dfrac{\partial^2 u}{dx\,dy} + y^2\dfrac{\partial^2 u}{dy^2} = 0$.

12. If $u = \dfrac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$, $\qquad \dfrac{\partial^2 u}{dx^2} + \dfrac{\partial^2 u}{dy^2} + \dfrac{\partial^2 u}{dz^2} = 0$.

13. If $u = e^{xyz}$, $\qquad \dfrac{\partial^3 u}{dx\,dy\,dz} = (1 + 3\,xyz + x^2y^2z^2)\,u.$

14. If $u = \sin^{-1}(xyz)$, $\qquad \dfrac{\partial^3 u}{dx\,dy\,dz} = \dfrac{1 + 2\,x^2y^2z^2}{(1 - x^2y^2z^2)^{\frac{5}{2}}}.$

111. Exact Differentials. — An expression of the form,

$$M\,dx + N\,dy, \tag{1}$$

where M and N are functions of x and y, may or may not be the differential of some function of x and y; if it is, it is called an *exact differential.* Some simple expressions may be seen at once to be exact differentials; thus, $y\,dx + x\,dy$ is an exact differential, for it is recognized as the total differential of xy.

If $M\,dx + N\,dy$ is an inexact differential, no function $F(x, y)$ can be found the differentiation of which will give this differential; thus $y\,dx - x\,dy$ is an inexact differential.

In applying the Calculus to problems in physics and mechanics expressions like (1) frequently arise and some test is needed to determine whether the expression can be gotten by the differentiation of any function of the variables involved.

As an example, the work W of moving a particle in the XY plane gives rise to the expression

$$dW = X\,dx + Y\,dy, \tag{2}$$

where X and Y are respectively the x- and y-components of the force acting on the particle. Since work is the product of force by distance,

$$X = \frac{\partial W}{dx} \quad \text{and} \quad Y = \frac{\partial W}{dy},$$

and (2) takes the form

$$dW = \frac{\partial W}{dx}\,dx + \frac{\partial W}{dy}\,dy. \tag{3}$$

Here (3) was not gotten by differentiation of any function $W = f(x, y)$ and the question is whether it could be so

gotten. In general, if M and N are any chosen functions of x and y, does a function of the independent variables (x, y) exist that will upon differentiation give $M\,dx + N\,dy$?

If there is such a function $u = F(x, y)$, then

$$du = \frac{\partial u}{dx} dx + \frac{\partial u}{dy} dy. \tag{4}$$

Now if the differentiation of the given function gives

$$M\,dx + N\,dy, \tag{1}$$

a comparison with the exact differential given in (4) gives

$$M = \frac{\partial u}{dx}, \qquad N = \frac{\partial u}{dy}; \tag{5}$$

that is, M and N must be the partial derivatives of the function u with respect to x and y, respectively.

According to the theorem of Art. 110, differentiating (5) gives

$$\frac{\partial M}{dy} = \frac{\partial^2 u}{dx\,dy} = \frac{\partial N}{dx}.$$

Hence, if $M = \dfrac{\partial u}{dx}$ and $N = \dfrac{\partial u}{dy}$, it is manifest that

$$\frac{\partial M}{dy} = \frac{\partial N}{dx} \tag{6}$$

is the necessary condition that $M\,dx + N\,dy$ may be gotten by the differentiation of a function $F(x, y)$, and it may be shown that it is a sufficient condition.

When the condition (6) is satisfied, $M\,dx + N\,dy$ is an *exact differential;* when the condition is not satisfied, $M\,dx + N\,dy$ is an *inexact differential.*

Example 1. — Given $M\,dx + N\,dy = y\,dx + x\,dy$.

Here $\quad M = y, \quad N = x, \quad \dfrac{\partial M}{dy} = 1, \quad \dfrac{\partial N}{dx} = 1.$

The condition (6) is satisfied and $y\,dx + x\,dy$ is an exact differential. The test is hardly needed in this simple case,

as it may be seen at once that the function sought is $xy+C$, where C is a constant, positive or negative, or zero.

Example 2. — Given $M\,dx + N\,dy = y\,dx - x\,dy$.

Here, since $\dfrac{\partial M}{dy} = 1$ and $\dfrac{\partial M}{dx} = -1$, the condition (6) is not satisfied; hence, $y\,dx - x\,dy$ is an inexact differential, and no function of (x, y) exists, the differentiation of which will give this differential.

Note. — If the equation $y\,dx - x\,dy = 0$ is given, it may be changed to an *exact differential equation;* $M\,dx + N\,dy = 0$, being called an *exact differential equation* when $M\,dx + N\,dy$ is an exact differential.

Thus, multiplying by y^{-2}, the equation given becomes

$$\frac{y\,dx - x\,dy}{y^2} = 0,$$

which is exact, and the function $F(x, y)$ is given by $x/y = C$. Again, multiplied by $1/xy$, the equation given becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0,$$

which is exact, and the function $F(x, y)$ is given by $\log x/y = \log C$. Either of these results evidently implies the other. Multiplying the equation given by $-x^{-2}$ gives $F(x, y)$ by $y/x = C_1$.

Example 3. — Given $M\,dx + N\,dy = \dfrac{y}{x}dx + \log x\,dy$.

$$\text{Here}\qquad M = \frac{y}{x},\quad N = \log x,\quad \frac{\partial M}{dy} = \frac{1}{x},\quad \frac{\partial N}{dx} = \frac{1}{x}.$$

The condition (6) is satisfied and the differential is exact. It is easy to recognize that $F(x, y)$ is in this case $y \log x$.

Example 4. — Given $M\,dx + N\,dy = \sin y\,dx + x\cos y\,dy$.

Here $M = \sin y$, $N = x\cos y$, $\dfrac{\partial M}{dy} = \cos y$, $\dfrac{\partial N}{dx} = \cos y$.

The condition (6) is satisfied and the differential is exact.

The function $F(x, y)$ may be seen to be $x \sin y$.

Example 5. — Given $x\,dy - y\,dx$. Change to value in polar coördinates, by $x = \rho \cos\theta$, $y = \rho \sin\theta$; $x\,dy - y\,dx = \rho^2\,d\theta$. Dividing by x^2,

$$\frac{x\,dy - y\,dx}{x^2} = \frac{\rho^2\,d\theta}{\rho^2 \cos^2\theta} = \sec^2\theta\,d\theta,$$

where the differentials are exact, and the function $F(x, y)$ is $y/x = \tan\theta$. (See Note Example 2.)

EXERCISE XX.

Determine which of the following differentials are exact, and for such as are exact find the functions that differentiated would give them:

1. $y \sin 2x\,dx + \sin^2 x\,dy$. *Ans.* $y \sin^2 x$.
2. $(ye^x + e^y)\,dx + (e^x + xe^y)\,dy$. *Ans.* $ye^x + xe^y$.
3. $(y^3 - 2xy)\,dx + (3xy^2 - x^2)\,dy$. *Ans.* $y^3x - x^2y$.
4. $v^n\,dp + npv^{n-1}\,dv$. *Ans.* pv^n.
5. $e^x \sin y\,dx + e^x \cos y\,dy$. *Ans.* $e^x \sin y$.
6. $\frac{y^2}{x}\,dx + x \log x\,dy$. *Ans.* Inexact.
7. $(x^2 - y)\,dx - x\,dy$. *Ans.* $\frac{1}{3}x^3 - xy$.
8. $e^x(x^2 + y^2 + 2x)\,dx + 2e^xy\,dy$. *Ans.* $e^x(x^2 + y^2)$.

112. Exact Differential Equations. — Equations of the form

$$M\,dx + N\,dy = 0,$$

are called exact differential equations when $M\,dx + N\,dy$ is an exact differential, the total differential of some function of (x, y), M and N being functions of x and y. The solving of differential equations involves the Integral Calculus, and the preceding Article with the Examples and Exercise are introductory to the subject.

The finding of the function from which an exact differential may be gotten by differentiation is essentially *Integration*, the inverse process to *Differentiation*.

In *Applied Calculus*, Part II, on the Integral Calculus, the subject of differential equations is given further treatment.

PART II.

INTEGRAL CALCULUS.

CHAPTER I.

INTEGRATION. STANDARD FORMS.

113. Inverse of Differentiation. — It has been shown that, when a function is given and its rate of change is required, the derivative, which expresses the rate of change, is gotten by the differentiation of the function.

It often occurs that the rate of change of a function is known and the value of the function is desired. In many problems in pure and applied mathematics the derivative or the differential of some function is given and the function itself is required.

The derivative or the differential of a function being given, it is a natural inference that an inverse operation to differentiation should yield the function. This inverse operation, the opposite of differentiation, is called *integration* and to *integrate* any given function (which when continuous is always the derivative of some other function) means to find that *other* function whose derivative is the *given* function. The function to be found is called an *integral* of the given function, which is called the *integrand;* that is, a function is an integral of its differential. The process of finding an integral of a given function is *integration, the inverse of differentiation;* that is, integration is anti-differentiation and an integral is an anti-differential.

When $dy = d(f(x))$; $d^{-1}(dy) = d^{-1}(df(x))$, (read "*the anti-differential of dy equals the anti-differential of d (f (x))*" is the inverse expression, reducing to $y = f(x)$, as the two symbols neutralize each other. The sign of integration is, however, $\int$, an elongated S; and this symbol indicates that the differential expression before which it appears is to be integrated, the whole expression denoting the integral itself.

Thus $\int dy \equiv d^{-1}(dy)$ and $\int d(f(x)) \equiv d^{-1}(df(x))$;

the sign of integration and the symbol of differentiation indicating inverse operations here neutralize each other, so

$$\int d(f(x)) = f(x).$$

There is here a close analogy with the algebraic signs of evolution and involution; for example, $\sqrt{x^2} = x$, the two symbols indicating inverse operations neutralizing each other. The analogy extends further to the fact that, while the operation of raising a given number to the second or other power is a direct operation and involves no difficulty in any case, the inverse operation of extracting a root may not be done so directly and in many cases can be done approximately only. While it has been shown that every continuous function has an integral,* this integral may not be expressible in terms of the elementary functions. In such cases, however, an approximate expression for the integral may be obtained by infinite series or by the measurement of an area representing the integral. Most of the functions that occur in practice can be integrated in terms of elementary functions, either directly by the knowledge acquired from differentiation, by reversing the rules of differentiation, or by reference to a table of integrals.

* By Picard, in *Traité d'Analyse.*

Except for simple differential expressions the process of integration is less simple and easy than the process of differentiation. Just as any finite number can be raised to a power, so can any finite continuous function be differentiated; and as the roots of some numbers can be expressed approximately only, so the integrals of some functions can be expressed approximately only.

There is *one* function whose integral is not some *other* function but is the function itself. This is the function e^x, whose derivative is e^x. As $\int e^x\,dx = e^x$, so $\sqrt{1} = 1$; the particular analogy in this exceptional case is manifest.

114. Indefinite Integral. — When

$$\frac{dy}{dx} = f'(x) \quad \text{or} \quad dy = f'(x)\,dx, \quad y = \int f'(x)\,dx,$$

read "*y is equal to an integral of* $f'(x)\,dx$." An integral of dy is evidently y, and $f(x)$ is an integral of its differential $f'(x)\,dx$.

Thus integrals of many simple differential expressions are known directly, by merely recalling the function which differentiated results in the given expression.

However, since the differential of any constant term of a function is zero, the function sought may contain a constant or constants no indication of which appears in the given differential or derivative.

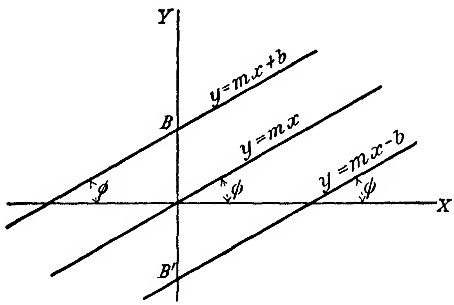
Hence, the integral of a differential expression is in general *indefinite,* owing to the lack of knowledge as to the existence or value, if existent, of constant terms of the function sought.

If $F(x)$ is a function whose derivative is $f(x)$, then $\int f(x)\,dx = F(x) + C$, is the *indefinite integral,* where C is a general constant, called the *constant of integration,* denoting a value either positive or negative or zero.

115. Illustrative Examples. — *Example* 1. — When $\frac{dy}{dx} = f'(x) = m$, is given as the constant slope of $y = f(x)$; then $y = \int m\,dx = mx + C$. The result is indefinite because $\frac{dy}{dx} = m$, is the slope of $y = mx + C$, $y = mx - C$, or $y = mx$. The constant C added to right member of the equation includes all constant terms, if any, of the function; for if the result be written $y + C' = mx + C''$, then $y = mx + C'' - C' = mx + C$. The letter C, often omitted, should be written as part of the result of the integration.

Data may be available in some cases to make the value of C known, or to eliminate it, and thus to make the result determinate.

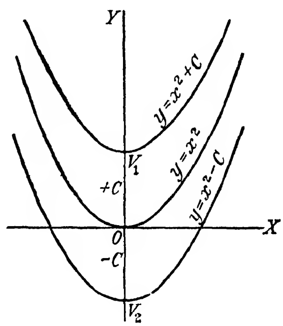
In this example, if it is known that the function has the value b when x is zero, then $y = mx + b$, since C is equal to b when x is zero. If y is $-b$, or if $y = 0$, when $x = 0$; then $y = mx - b$, or $y = mx$.



As shown in the figure the function is a straight line making an angle ϕ ($= \tan^{-1} m$) with the X-axis, the constant of integration being the Y-intercept. The *indefinite* or *general integral* is $y = mx + C$, any straight line with slope m.

Note: — When the value of C is determined the integral is called a *particular* integral.

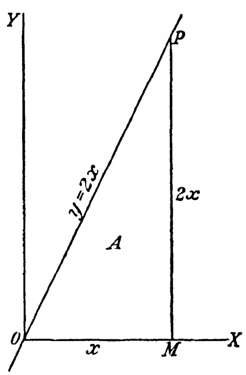
Example 2. — When $\frac{dy}{dx} = 2x$ is given as the rate of change of y with respect to x, then $y = \int 2x\,dx = x^2 + C$, where $x^2 + C$ is the general integral of $2x\,dx$, since the differential of $(x^2 + C)$ is $2x\,dx$. Here $x^2 + C$ is a function whose rate of change is $2x$, and if the rate of change is that of the ordinate to the abscissa, or the slope of a curve, then the integral, $y = x^2 + C$, is the equation of the curve. The locus of the equation is a parabola with its vertex at a distance C above or below the origin, or at the origin, according as the value of C is positive, negative, or zero.

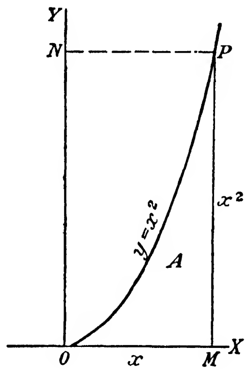


If y is known for some value of x, then the value of C is easily determined. For instance, if it is known that the point (a, b) lies on the curve, then $y = x^2 + C$, must be satisfied by the coördinates (a, b), giving $b = a^2 + C$, and, therefore, $C = b - a^2$. Hence, the particular parabola is $y = x^2 + b - a^2$. If the curve is known to pass through the origin, then, since C is zero, $y = x^2$ is the parabola with vertex at the origin.

Example 3. — If a given derivative $\frac{dA}{dx} = 2x$ represents the rate of change of an area A to a length x, then $A = \int 2x\,dx = x^2 + C$, is an area where x^2 may represent the area of a variable triangle formed by the straight line $y = 2x$, the ordinate at any value of x, and X-axis. In the figure

shown the area A is zero when x is zero, and therefore C is zero. The area of any triangle being one-half the product of base and altitude, the result of the integration, $A = x^2$, is seen to be true. The area $A = x^2 + C$ may represent a square of side x and some additional area represented by C, undetermined, as it might be positive, negative, or zero — the derivative of the area in either case being the given rate $2x$.





Example 4. — If the given derivative is $\dfrac{dA}{dx} = x^2$, then

$$A = \int x^2\,dx = \frac{x^3}{3} + C, \quad \text{since} \quad d\left(\frac{x^3}{3} + C\right) = x^2\,dx.$$

Here the area $\dfrac{x^3}{3}$ is that bounded by the curve, a parabola $y = x^2$, the ordinate at any value of x, and the X-axis. In the figure shown the area A is zero when x is zero, and therefore C is zero.

It is seen that the area OPM is exactly one-third of the area of the circumscribed rectangle. Hence, the area OPN between the curve, the abscissa at the end of any ordinate, and the Y-axis, is two-thirds the area of the same rectangle.

Example 5. — The acceleration of a falling body being nearly constant near the earth's surface, it is required to find the velocity and the distance after any time. If s denotes the distance along a straight line positive upward and t the time, then $\frac{d^2s}{dt^2}$ is the acceleration.

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = -g \quad \text{or} \quad d\left(\frac{ds}{dt}\right) = -g\,dt. \tag{1}$$

Integrating gives velocity,

$$v = \frac{ds}{dt} = -gt + (C = v_0), \tag{2}$$

where v_0 is the initial velocity, when $t = 0$; then,

$$ds = -gt\,dt + v_0\,dt.$$

Integrating gives distance,

$$s = -\tfrac{1}{2}\,gt^2 + v_0t + (C = s_0), \tag{3}$$

where s_0 is the initial distance, when $t = 0$. If the body falls from rest, v_0 and s_0 are zero, hence; $v = -gt$; $s = -\frac{1}{2}\,gt^2$; and $v^2 = -2\,gs$, by eliminating t. (See Art. 14.)

Example 6. — Determine v and s in terms of t for a bullet shot vertically upward with a velocity of 2000 feet per second, neglecting air resistance.

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = -32.2 \text{ ft. per sec. per sec.}$$

$$v = \frac{ds}{dt} = \int\frac{d^2s}{dt} = \int -32.2\,dt = -32.2\,t + (C = v_0 = 2000),$$

$$v = v_0, \quad \text{when } t = 0.$$

$$s = \int ds = \int -32.2\,t\,dt + 2000\,dt = -16.1\,t^2 + 2000\,t$$
$$+ (C = s_0 = 0), \quad s = s_0 = 0, \quad \text{when } t = 0.$$

To find the time of rising, make $v = 0 = -32.2\,t + 2000$;

$$\therefore \quad t = 62.1 \text{ sec.}$$

To find the height it will rise, $s = -16.1(62.1)^2 + 2000(62.1) = 62{,}112$ ft.

To find the time of flight, $s = 0 = -16.1\,t^2 + 2000\,t$;

$$\therefore \quad t = 124.2 \text{ sec.} \quad \text{and} \quad t = 0.$$

Hence, the time of falling is the same as that of rising, since the time of flight is twice that of rising. The height it will rise may be found, by making $v = 0$ in

$$v^2 = v_0^2 - 2\,gs; \quad \therefore \quad s = \frac{v_0^2}{2\,g} = \frac{(2000)^2}{64.4} = 62{,}112 \text{ ft.},$$

the same as above.

Remarks. — These examples illustrate the important fact that the knowledge of the rate of change of a quantity together with the knowledge of its original value, makes possible the complete determination of the value of that quantity at any time. This must be so, since two different quantities with the same rate of change always have a constant difference, the rate of change of their difference being zero. This is in accordance with the undoubted fact that if the rate of change of a quantity decreases to zero and remains zero, the quantity ceases to change at all, being then constant. The fact is formulated in principle (iv) of Art. 116, and is the converse of the fact that if a quantity is constant, its rate is zero.

116. Elementary Principles. — While there is a general method of differentiation, for the inverse process of integration no general method has been devised. For the integration of the various differential expressions, rules have been formulated and special methods have been found, one or more of which provide for every case in which integration is possible.

These rules or formulas are derived or disclosed through knowledge of the rules of differentiation; in fact, the rules most used are merely directions for retracing the steps taken in differentiation.

Elementary principles that apply in integration may be expressed as follows:

$$\text{(i)} \quad \int f(x)\,dx \equiv F(x) + C, \quad \text{if} \quad dF(x) \equiv f(x)\,dx.$$

This principle furnishes the most direct proof of formulas for indefinite integration, and provides a decisive test of the correctness of the result of any integration. Thus,

$$\int x^n\,dx \equiv \frac{x^{n+1}}{n+1} + C, \qquad \text{since} \quad d\left(\frac{x^{n+1}}{n+1}\right) \equiv x^n\,dx;$$

$$\int \frac{dx}{x} \equiv \log x + C, \qquad \text{since} \quad d(\log x) \equiv \frac{dx}{x};$$

$$\int b^x\,dx \equiv \frac{b^x}{\log b} + C, \qquad \text{since} \quad d\left(\frac{b^x}{\log b}\right) \equiv b^x\,dx.$$

In this manner the test can be applied to prove any formula, or to verify the result of the integration of any expression.

(ii) *A constant factor can be transposed from one side of the sign of integration to the other, and a constant factor can be introduced on one side, if its reciprocal is introduced on the other, without changing the value of the integral.*

For, if a is a constant,

$$\int ay\,dx \equiv a \int y\,dx,$$

since the differentiation of the equation gives, $ay\,dx \equiv ay\,dx$. Hence,

$$\int y\,dx \equiv \frac{1}{a} \int ay\,dx \equiv a \int \frac{1}{a} y\,dx.$$

(iii) *The integral of a polynomial is equal to the sum of the integrals of its several terms.* For

$$\int (a + x - x^2)\,dx \equiv \int a\,dx + \int x\,dx - \int x^2\,dx,$$

since the differentiation of the equation gives

$$a\,dx + x\,dx - x^2\,dx \equiv a\,dx + x\,dx - x^2\,dx.$$

(iv) $\int 0 \equiv C, \quad \text{since} \quad dC \equiv 0.$

The integral of zero is a constant.

Thus, if $\dfrac{ds}{dt} = v$, where v is constant velocity,

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = 0; \text{ that is, acceleration is zero,}$$

hence,
$$\int \frac{dv}{dt} = \int 0 = C = v.$$

117. Standard Forms and Formulas.* — There follows a list of standard *integrable* forms, that is, differential functions whose integrals can be expressed in finite forms involving no other than algebraic, trigonometric, inverse trigonometric, exponential, or logarithmic functions. To integrate a function that is not expressed in terms of an immediately integrable form, it is reduced if possible to one or more of such forms and the formula applied.

The formulas in general are gotten by merely reversing the formulas for differentiation, and each can be proved by the principle (i) of Art. 116.

The list will be found to contain the one or more than one integral to which every integrable form is reducible. These forms may, therefore, be called *fundamental* although only the first three are really fundamental, since each of the others by substitutions can be reduced to one of the three. While the list is of *standard* integrable forms, it may be supplemented by other integrable forms; but no list of forms is exhaustive, even when extended into tables of integrals. After the acquirement of familiarity with the rules of differentiation, and the common methods of reduction with the standard integrals, the use of tables of integrals for the complicated forms is recommended; much time otherwise given to formal work in integration being thereby saved.

* See Appendix.

different forms, but upon reduction they will always be found to differ (if at all) only by a constant, in accordance with the statement above.

Formula I is the standard formula for the *Power Form.* It is of most frequent application, and it may be expressed in words as follows:

The integral of the product of a variable base with any constant exponent (except -1*) and the differential of the base is the base, with its exponent increased by* 1, *divided by the increased exponent, and a constant.*

The proof has been given in (i), Art. 116; it may be derived thus: since

$$\int 2\,x\,dx = x^2 + C, \quad \int 3\,x^2\,dx = x^3 + C, \text{ etc.,}$$

$$\int (n+1)\,x^n\,dx = x^{n+1} + C;$$

hence, in general,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

When a given integrand is a fraction with denominator to a power, it may become this form by bringing up the variable quantity with change of exponent's sign; but, since the variable quantity is represented by x in the formula, it is essential that the differential of the variable quantity, and not merely dx, be present in the integrand before the formula is applicable.

If a constant factor is lacking, it may be supplied in accordance with (ii), Art. 116; but it should be noted that the principle is only for *constant* factors.

The value of an integral is changed when a *variable* factor is transferred from one side of the sign $\int$ to the other; thus,

$$\int x^2\,dx = \tfrac{1}{3}\,x^3 + C, \quad \text{but} \quad x\int x\,dx = \tfrac{1}{2}\,x^3 + C.$$

When a change of sign is needed, the constant factor -1 effects the change. Thus, for an example,

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\frac{1}{2}\int (a^2 - x^2)^{-\frac{1}{2}}(-2x\,dx) = -\sqrt{a^2 - x^2} + C.$$

To verify:

$$\int d(-\sqrt{a^2 - x^2} + C) = \int -\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x\,dx)$$

$$= \int \frac{x\,dx}{\sqrt{a^2 - x^2}}, \text{ as given.}$$

When a variable factor is lacking, resort may be had to expansion and then application of the formula to each term of the polynomial. Thus, for an example:

$$\int 2(1+x^2)^2\,dx = 2\int (1+2x^2+x^4)\,dx = 2(x+\tfrac{2}{3}x^3+\tfrac{1}{5}x^5)+C.$$

When the numerator is of higher power than denominator, reduce by division and then apply formula or formulas, thus:

$$\int \frac{x^2+1}{x-1}\,dx = \int \left(x + 1 + \frac{2}{x-1}\right)dx$$

$$= \tfrac{1}{2}x^2 + x + 2\log(x-1) + C.$$

If $n = -1$, formula I gives a result that is not finite; but when $n = -1$, the form reduces to form II and that formula applies. Thus,

$$\int x^{-1}\,dx = \int \frac{dx}{x} = \log x + C.$$

Formula II may be stated in words as follows:

The integral of a fraction whose numerator is the differential of its denominator is the Napierian logarithm of the denominator, and a constant. The result will be *real* only when x is positive. When

$$x > a, \quad \int \frac{dx}{x-a} = \log(x-a) + C;$$

but, if

$$x < a, \quad \int \frac{dx}{x-a} = \int \frac{-dx}{a-x} = \log(a-x) + C.$$

Formula III may be stated in words as follows:

The integral of the product of a constant base with a variable exponent and the differential of the exponent is the base, with exponent unchanged, divided by the Napierian logarithm of the base, and a constant.

Here the base b must be positive and not unity.

Formula IV is the special case of III, the Napierian logarithm of base e being unity.

In applying these two formulas to given integrands, it is essential that the differential of the variable quantity, and not merely dx, be present. Thus, for examples:

$$\int b^{x^4}x^3\,dx = \frac{1}{4\log b}\int b^{x^4}\log b\,4\,x^3\,dx = \frac{b^{x^4}}{4\log b} + C.$$

$$\int e^{x/n}\,dx = n\int e^{x/n}\frac{dx}{n} = ne^{x/n} + C.$$

$$\int \pi^{3x}\,dx = \frac{1}{3\log\pi}\int \pi^{3x}\log\pi\cdot 3\,dx = \frac{\pi^{3x}}{3\log\pi} + C.$$

The following are examples of integration by one or more of the first four standard formulas.

EXERCISE XXI.

In these examples the results may be verified by (i), Art. 116, and the verification should be made where the result is not given.

1. $\int\left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}} + \frac{2}{x^5} - \frac{2}{x}\right)dx = \frac{3\,x^{\frac{5}{3}}}{5} - 3\,x^{\frac{1}{3}} - \frac{1}{2\,x^4} - 2\log x + C.$

2. $\int (ax+b)^n\,dx = \frac{1}{a}\int (ax+b)^n\,a\,dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + C.$

3. $\int (2\,a + 3\,bx)^3\,dx.$

4. $\int \frac{x^2\,dx}{(a^2+x^3)^{\frac{1}{2}}}.$

5. $\int \left(1 + \frac{9\,x}{4}\right)^{\frac{1}{2}} dx.$

6. $\int t^2\,(2-t^3)^3\,dt.$

7. $\int 2\,\pi y\left(\frac{y^2}{p^2}+1\right)^{\frac{1}{2}} dy.$

8. $\int \sqrt{2\,s+1}\,ds.$

9. $\int \sqrt{2\,px}\,dx = \sqrt{2\,p}\int x^{\frac{1}{2}}\,dx = \frac{2}{3}\sqrt{2\,p}x^{\frac{3}{2}}\left(= \frac{2}{3}\,x\sqrt{2\,px}\right) + C.$

10. $\int (ax^n+b)^p\,x^{n-1}\,dx = \frac{1}{na}\int (ax^n+b)^p\,nax^{n-1}dx = \frac{(ax^n+b)^{p+1}}{na\,(p+1)} + C.$

11. $\int 5x\sqrt{1-2x^2}\,dx = -\frac{5}{4}\int (1-2x^2)^{\frac{1}{2}}(-4x\,dx) = -\frac{5}{6}(1-2x^2)^{\frac{3}{2}} + C.$

12. $\int \sqrt{1-e^x}\,e^x\,dx = -\int (1-e^x)^{\frac{1}{2}}(-e^x\,dx) = -\frac{2}{3}(1-e^x)^{\frac{3}{2}} + C.$

13. $\int \frac{a\,dx}{x^n} = a\int x^{-n}\,dx = \frac{ax^{1-n}}{1-n} + C.$

14. $\int \frac{dx}{\sqrt{x}}.$

15. $\int \sqrt{x}\,(1-x^2)\,dx.$

16. $\int \frac{x^{n-1}}{a+bx^n}\,dx = \frac{1}{nb}\int \frac{nbx^{n-1}}{a+bx^n}\,dx = \frac{1}{nb}\log(a+bx^n) + C.$

17. $\int \frac{-(2ax-x^2)\,dx}{(3ax^2-x^3)^{\frac{1}{3}}} = \frac{-(3ax^2-x^3)^{\frac{2}{3}}}{2} + C.$

18. $\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x}\,dx = \frac{1}{2}\log(x^2+2x) + C.$

19. $\int \frac{5(2a-x^2)^3}{x^5}\,dx = 5\left(-\frac{2a^3}{x^4} + \frac{6a^2}{x^2} + 6a\log x - \frac{x^2}{2}\right) + C.$

20. $\int (\log x)^n\frac{dx}{x} = \frac{(\log x)^{n+1}}{n+1} + C.$

21. $\int (\log x)^3\frac{dx}{x}.$

22. $\int \frac{x^3}{x+1}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(x+1) + C.$

23. $\int \frac{dx}{x\log x} = \log(\log x) + C.$

24. $\int \frac{3x+1}{3x-1}\,dx = x + \log(3x-1)^{\frac{2}{3}} + C.$

25. $\int \frac{dx}{e^x(3+e^{-x})} = \int \frac{e^{-x}\,dx}{3+e^{-x}}.$

26. $\int \frac{\sin 2x}{1+\sin^2 x}\,dx.$

27. $\int \frac{\cot x}{\log\sin x}\,dx.$

28. $\int (e^x-e^{-x})^2\,dx = \frac{(e^{2x}-e^{-2x})}{2} - 2x + C.$

29. $\int \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)dx.$

30. $\int \frac{e^x-2}{e^x+2}\,dx = \int\left(-dx + \frac{2e^x\,dx}{e^x+2}\right) = 2\log(e^x+2) - x + C.$

31. $\int \frac{e^{2x}}{e^x+1}\,dx.$

32. $\int a^x b^x\,dx = \frac{a^x b^x}{\log a + \log b} + C.$

33. $\int \frac{\cos 2x}{\sin 2x}\,dx.$

34. $\int \frac{\sqrt{x}}{1+x^{\frac{3}{2}}}\,dx.$

35. $\int \frac{\log x\,dx}{x(1-\log^2 x)} = -\frac{1}{2}\int \frac{-2\log x}{1-\log^2 x}\frac{dx}{x}.$

36. $\int \frac{dx}{(1+x^2)\arctan x} = \int \frac{dx/1+x^2}{\tan^{-1}x}.$

119. Derivation of Formulas XI, XII, XIII, and XIV. — By the application of Formula II the following results:

$$\int \tan x\,dx = \int \frac{\tan x \sec x\,dx}{\sec x}$$

$$= \log \sec x + C = -\log \cos x + C. \qquad \text{(XI)}$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$$

$$= \log \sin x + C = -\log \csc x + C. \qquad \text{(XII)}$$

$$\int \csc x\,dx = \int \frac{dx}{\sin x}$$

$$= \int \frac{dx}{2 \sin x/2 \cos x/2}$$

$$= \int \frac{\sec^2 x/2\,dx/2}{\tan x/2} = \log \tan x/2 + C. \qquad \text{(XIII)}$$

Or

$$\int \csc x\,dx = \int \frac{\csc x\,(-\cot x + \csc x)\,dx}{\csc x - \cot x}$$

$$= \log(\csc x - \cot x) + C.$$

$$\int \sec x\,dx = \int \csc(x + \pi/2)\,dx$$

$$= \log \tan(x/2 + \pi/4) + C. \qquad \text{(XIV)}$$

Or

$$\int \sec x\,dx = \int \frac{\sec x\,(\tan x + \sec x)\,dx}{\sec x + \tan x}$$

$$= \int \frac{\sec x \tan x\,dx + \sec^2 x\,dx}{\sec x + \tan x}$$

$$= \log(\sec x + \tan x) + C.$$

EXERCISE XXII.

By Art. 116, and one or more of the standard formulas I to XIV.

1. $\int\left(\sin 3x + \cos 5x - \sin\frac{x}{2}\right)dx = -\frac{\cos 3x}{3} + \frac{\sin 5x}{5} + 2\cos\frac{x}{2} + C.$

2. $\int \sin x \cos x\,dx = \frac{\sin^2 x}{2} + C.$

3. $\int \cos x \sin x\,dx = -\frac{\cos^2 x}{2} + C.$

4. $\int\left(\frac{\sin\theta}{\cos^2\theta}+\frac{\cos\theta}{\sin^2\theta}\right)d\theta = -\int\cos^{-2}\theta\,(-\sin\theta\,d\theta)+\int\sin^{-2}\theta\cos\theta\,d\theta.$

5. $\int\frac{\cos^2\theta}{\sin\theta}\,d\theta = \int\frac{(1-\sin^2\theta)}{\sin\theta}\,d\theta = \int(\csc\theta-\sin\theta)\,d\theta.$

6. $\int\sin^2\theta\,d\theta = \int\frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\theta-\frac{1}{4}\sin(2\theta)+C.$

7. $\int\cos^2\theta\,d\theta = \int\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+C.$

8. $\int\sin^3\theta\,d\theta = \int(1-\cos^2\theta)\sin\theta\,d\theta = -\cos\theta+\frac{1}{3}\cos^3\theta+C.$

9. $\int\sin^3\theta\cos^3\theta\,d\theta = \int\sin^3\theta\,(1-\sin^2\theta)\,d\sin\theta = \frac{1}{4}\sin^4\theta-\frac{1}{6}\sin^6\theta+C.$

10. $\int\tan^2 x\,dx = \int(\sec^2 x-1)\,dx = \tan x - x + C.$

11. $\int\cot^2 x\,dx = \int(\csc^2 x-1)\,dx = -\cot x - x + C.$

12. $\int\tan^3\theta\,d\theta = \int\tan\theta\,(\sec^2\theta-1)\,d\theta = \frac{1}{2}\tan^2\theta-\log\sec\theta+C$
$= \frac{1}{2}\tan^2\theta+\log\cos\theta+C.$

13. $\int\sec^2(ax^2)\,x\,dx = \frac{1}{2a}\tan(ax^2)+C.$

14. $\int\frac{\tan(ax^2)}{(\cos(ax^2))}x\,dx = \frac{1}{2a}\int\sec(ax^2)\tan(ax^2)\,d(ax^2) = \frac{1}{2a}\sec(ax^2)+C.$

15. $\int\frac{dx}{\sin x\cos x} = \int\frac{2\,dx}{\sin(2x)} = \int\csc(2x)\,d(2x)$
$= \log\tan x + C$, by XIII.

16. $\int\frac{dx}{\sin x\cos x} = \int\csc x\sec x\,dx = \int\frac{\sec^2 x}{\tan x}\,dx = \log\tan x + C$, by II.

17. $\int\sec^2\theta\csc^2\theta\,d\theta = \int(\sec^2\theta+\csc^2\theta)\,d\theta = \tan\theta-\cot\theta+C.$

18. $\int\frac{\cot\theta+\tan\theta}{\cot\theta-\tan\theta}\,d\theta = \int\sec 2\theta\,d\theta = \frac{1}{2}\log\tan\left(\theta+\frac{\pi}{4}\right)+C.$

19. $\int\frac{1-\sin\theta}{1+\sin\theta}\,d\theta = \int\frac{(1-\sin\theta)^2}{\cos^2\theta}\,d\theta = 2(\tan\theta-\sec\theta)-\theta+C.$

20. $\int e^{\sin x}\cos x\,dx.$ 21. $\int e^{2\cos x}\sin x\,dx.$

22. $\int b^{\tan(ax)}\sec^2(ax)\,dx.$

120. Derivation of Formulas XVI, XVII, XIX, and XX. — By the application of Formula II, the following results:

For XVI, put

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right),$$

$$\therefore \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int \frac{dx}{a+x} - \frac{1}{2a}\int \frac{-dx}{a-x}$$

$$= \frac{1}{2a}\log(a+x) - \frac{1}{2a}\log(a-x) + C$$

$$= \frac{1}{2a}\log\frac{a+x}{a-x} + C$$

$$= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C'. \quad (x^2 < a^2) \qquad \text{(XVI)}$$

or

$$\int \frac{dx}{a^2 - x^2} = \int \frac{-dx}{x^2 - a^2} = \frac{1}{2a}\int \frac{dx}{x+a} - \frac{1}{2a}\int \frac{dx}{x-a}$$

$$= \frac{1}{2a}\log\frac{x+a}{x-a} + C = \frac{1}{a}\coth^{-1}\frac{x}{a} + C'. \quad (x^2 > a^2)$$

The first or second of these results is used according as $a - x$ or $x - a$ is positive; that is, the form of the result which is *real* is to be taken.

For XVII, put

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right),$$

$$\therefore \quad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a}$$

$$= \frac{1}{2a}\log(x-a) - \frac{1}{2a}\log(x+a) + C$$

$$= \frac{1}{2a}\log\frac{x-a}{x+a} + C$$

$$= -\frac{1}{a}\coth^{-1}\frac{x}{a} + C', \quad (x^2 > a^2) \qquad \text{(XVII)}$$

or

$$\int \frac{dx}{x^2 - a^2} = \int \frac{-dx}{a^2 - x^2} = \frac{1}{2a}\int \frac{-dx}{a-x} - \frac{1}{2a}\int \frac{dx}{a+x}$$

$$= \frac{1}{2a}\log\frac{a-x}{a+x} + C = -\frac{1}{a}\tanh^{-1}\frac{x}{a} + C'. \quad (x^2 < a^2)$$

The first or second of these results is used according as $x - a$ or $a - x$ is positive.

For XIX, let

$$\sqrt{x^2 + a^2} = z - x; \quad \text{or} \quad z = x + \sqrt{x^2 + a^2}, \tag{1}$$

$$\therefore \quad a^2 = z^2 - 2\,xz.$$

$$d\,(a^2) = 0 = 2\,z\,dz - 2\,x\,dz - 2\,z\,dx;$$

$$(z - x)\,dz = z\,dx;$$

$$\frac{dz}{z} = \frac{dx}{z - x} = \frac{dx}{\sqrt{x^2 + a^2}}, \quad \text{by (1)}$$

$$\therefore \quad \int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{dz}{z} = \log z + C = \log\,(x + \sqrt{x^2 + a^2}) + C,$$

or

$$\sinh^{-1} \frac{x}{a} + C'. \tag{XIX}$$

For XX, similarly, on letting $\sqrt{x^2 - a^2} = z - x$,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\,(x + \sqrt{x^2 - a^2}) + C, \text{ or } \cosh^{-1} \frac{x}{a} + C'. \tag{XX}$$

The logarithmic form of $\cosh^{-1} \frac{x}{a}$ is $\log \left(\frac{x + \sqrt{x^2 - a^2}}{a} \right)$, but its derivative or differential is the same as that of $\log\,(x + \sqrt{x^2 - a^2})$, the constant a disappearing in the differentiation; and so too with the $\sinh^{-1} \frac{x}{a}$. (See Art. 66.)

121. Derivation of Formulas XV, XVIII, XXI, and XXII. — These formulas are merely the reverse of the differential forms given in Examples 1 and 2, Exercise VI.

They may be derived from the forms for the inverse trigonometric functions of x. Thus:

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \int \frac{\dfrac{dx}{a}}{1 + \dfrac{x^2}{a^2}} = \frac{1}{a} \int \frac{d\left(\dfrac{x}{a}\right)}{1 + \left(\dfrac{x}{a}\right)^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C,$$

since

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x + C.$$

Since

$$\tan^{-1}\frac{x}{a} = \frac{\pi}{2} - \cot^{-1}\frac{x}{a}, \quad d\left(\tan^{-1}\frac{x}{a}\right) = d\left(-\cot^{-1}\frac{x}{a}\right).$$

Hence

$$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad \text{or} \quad -\frac{1}{a}\cot^{-1}\frac{x}{a} + C'. \qquad \text{(XV)}$$

In the same way the second forms follow for formulas XVIII, XXI, and XXII.

The standard forms are given in terms of $\frac{x}{a}$, because they are of more use than those in terms of x; the latter, being special cases where $a = 1$, are often given as the standard forms. Integrals may be obtained by reduction to either form.

EXERCISE XXIII.

1. $\displaystyle\int\frac{dx}{b^2+c^2x^2} = \frac{1}{c}\int\frac{d\,(cx)}{b^2+(cx)^2} = \frac{1}{bc}\tan^{-1}\frac{cx}{b} + C.$

2. $\displaystyle\int\frac{dx}{b^2-c^2x^2} = \frac{1}{c}\int\frac{d\,(cx)}{b^2-(cx)^2} = \frac{1}{2\,bc}\log\frac{b+cx}{b-cx} + C \quad (c^2x^2 < b^2)$

$$= \frac{1}{bc}\tanh^{-1}\frac{cx}{b} + C'$$

$$= \frac{1}{c}\int\frac{-d\,(cx)}{(cx)^2-b^2} = \frac{1}{2\,bc}\log\frac{cx+b}{cx-b} + C \quad (c^2x^2 > b^2)$$

$$= \frac{1}{bc}\coth^{-1}\frac{cx}{b} + C'.$$

3. $\displaystyle\int\frac{dx}{x^2+6x+12} = \int\frac{dx}{(x+3)^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+3}{\sqrt{3}} + C.$

4. $\displaystyle\int\frac{dx}{x^2+6x+5} = \int\frac{dx}{(x+3)^2-4} = \frac{1}{4}\log\frac{(x+3)-2}{(x+3)+2} + C$

$$= \frac{1}{4}\log\frac{x+1}{x+5} + C.$$

5. $\displaystyle\int\frac{x^2\,dx}{x^6-1} = \frac{1}{6}\log\frac{x^3-1}{x^3+1} + C.$

6. $\displaystyle\int\frac{dx}{9x^2-4}.$

7. $\displaystyle\int\frac{x\,dx}{x^4+a^4} = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2} + C.$

8. $\displaystyle\int\frac{dx}{16x^2+9}.$

9. $\displaystyle\int\frac{dx}{c^2x^2-b^2}=\frac{1}{2\,bc}\log\frac{cx-b}{cx+b}+C=-\frac{1}{bc}\coth^{-1}\frac{cx}{b}+C'.\quad (c^2x^2>b^2)$

$\displaystyle=\frac{1}{2\,bc}\log\frac{b-cx}{b+cx}+C=-\frac{1}{bc}\tanh^{-1}\frac{cx}{b}+C'.\quad (c^2x^2<b^2)$

10. $\displaystyle\int\frac{dx}{ax^2+bx+c}=2\int\frac{2\,a\,dx}{(2\,ax+b)^2+4\,ac-b^2}$

$\displaystyle=\frac{2}{\sqrt{4\,ac-b^2}}\tan^{-1}\frac{2\,ax+b}{\sqrt{4\,ac-b^2}}+C\quad (4\,ac>b^2)$

$\displaystyle=\frac{1}{\sqrt{b^2-4\,ac}}\log\frac{2\,ax+b-\sqrt{b^2-4\,ac}}{2\,ax+b+\sqrt{b^2-4\,ac}}+C.\quad (4\,ac<b^2)$

11. $\displaystyle\int\frac{2\,x+7}{x^2+4\,x+5}dx=\int\frac{(2\,x+4)\,dx}{x^2+4\,x+5}+3\int\frac{dx}{(x+2)^2+1}$

$\displaystyle=\log(x^2+4\,x+5)+3\tan^{-1}(x+2)+C.$

12. $\displaystyle\int\frac{x\,dx}{x^2+2\,x+1}=\int\frac{x+1}{(x+1)^2}dx-\int\frac{dx}{(x+1)^2}$

$\displaystyle=\log(x+1)+\frac{1}{x+1}+C.$

13. $\displaystyle\int\frac{dx}{\sqrt{a^2c^2-b^2x^2}}=\frac{1}{b}\int\frac{b\,dx}{\sqrt{(ac)^2-(bx)^2}}=\frac{1}{b}\sin^{-1}\frac{bx}{ac}+C,$

or $\displaystyle-\frac{1}{b}\cos^{-1}\frac{bx}{ac}+C'.$

14. $\displaystyle\int\frac{dx}{\sqrt{b^2x^2+a^2c^2}}=\frac{1}{b}\log\left(bx+\sqrt{b^2x^2+a^2c^2}\right)+C,$ or $\displaystyle\sinh^{-1}\frac{bx}{ac}+C'.$

15. $\displaystyle\int\frac{dx}{\sqrt{b^2x^2-a^2c^2}}=\frac{1}{b}\log\left(bx+\sqrt{b^2x^2-a^2c^2}\right)+C,$ or $\displaystyle\cosh^{-1}\frac{bx}{ac}+C'.$

16. $\displaystyle\int\frac{dx}{\sqrt{a-bx^2}}=\frac{1}{\sqrt{b}}\int\frac{dx}{\sqrt{a/b-x^2}}=\frac{1}{\sqrt{b}}\sin^{-1}x\sqrt{\frac{b}{a}}+C.$

17. $\displaystyle\int\frac{dx}{\sqrt{3-2\,x^2}}=\frac{1}{\sqrt{2}}\sin^{-1}x\sqrt{\frac{2}{3}}+C.$ 18. $\displaystyle\int\frac{dx}{\sqrt{3-4\,x^2}}.$

19. $\displaystyle\int\frac{dx}{\sqrt{1-x-x^2}}=\int\frac{dx}{\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}}=\sin^{-1}\frac{2\,x+1}{\sqrt{5}}+C.$

20. $\displaystyle\int\frac{dx}{\sqrt{2+2\,x-x^2}}=\sin^{-1}\frac{x-1}{\sqrt{3}}+C.$

21. $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{2\,a\,dx}{\sqrt{(2\,ax+b)^2+4\,ac-b^2}}$

$\displaystyle=\frac{1}{\sqrt{a}}\log\left(2\,ax+b+2\sqrt{a}\,\sqrt{ax^2+bx+c}\right)+C.$

22. $\displaystyle\int\frac{dx}{\sqrt{x^2+2\,ax}}=\log\left(x+a+\sqrt{x^2+2\,ax}\right)+C.$

23. $\int \frac{dx}{\sqrt{ax^2 - b}} = \frac{1}{\sqrt{a}} \log (x\sqrt{a} + \sqrt{ax^2 - b}) + C.$

24. $\int \frac{dx}{\sqrt{-ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{2a\,dx}{\sqrt{4ac + b^2 - (2ax - b)^2}}$

$$= \frac{1}{\sqrt{a}} \sin^{-1} \frac{2ax - b}{\sqrt{4ac + b^2}} + C.$$

25. $\int \frac{\sqrt{1 + x}}{\sqrt{1 - x}} dx = \int \frac{1 + x}{\sqrt{1 - x^2}} dx = \int \frac{dx}{\sqrt{1 - x^2}} + \int (1 - x^2)^{-\frac{1}{2}} x\,dx.$

$$= \sin^{-1} x - \sqrt{1 - x^2} + C.$$

26. $\int \frac{\sqrt{x + 1}}{\sqrt{x - 1}} dx = \int \frac{(x + 1)\,dx}{\sqrt{x^2 - 1}} = \sqrt{x^2 - 1} + \log (x + \sqrt{x^2 - 1}) + C.$

27. $\int \frac{dx}{x\sqrt{c^2x^2 - a^2b^2}} = \int \frac{c\,dx}{cx\sqrt{(cx)^2 - (ab)^2}} = \frac{1}{ab} \sec^{-1} \frac{cx}{ab} + C.$

28. $\int \frac{5\,dx}{x\sqrt{3x^2 - 5}}.$

29. $\int \frac{\sqrt{x+a}}{x\sqrt{x-a}} dx = \int \frac{x+a}{x\sqrt{x^2-a^2}} dx = \sec^{-1} \frac{x}{a} + \log (x + \sqrt{x^2 - a^2}) + C.$

30. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \int \frac{x^2 - a^2}{x\sqrt{x^2 - a^2}} dx = \int \frac{x\,dx}{\sqrt{x^2 - a^2}} - \int \frac{a^2\,dx}{x\sqrt{x^2 - a^2}}$

$$= \sqrt{x^2 - a^2} - a \sec^{-1} \frac{x}{a} + C.$$

31. $\int \frac{dx}{(1 - x^2)^{\frac{3}{2}}} = \int \frac{x^{-3}\,dx}{(x^{-2} - 1)^{\frac{3}{2}}} = \int (x^{-2} - 1)^{-\frac{3}{2}} x^{-3}\,dx = \frac{x}{\sqrt{1 - x^2}} + C.$

32. $\int \frac{dx}{\sqrt{2abx - b^2x^2}} = \frac{1}{b} \int \frac{b\,dx}{\sqrt{2a(bx) - (bx)^2}} = \frac{1}{b} \operatorname{vers}^{-1} \frac{bx}{a} + C.$

33. $\int \frac{-dx}{\sqrt{8bx - b^2x^2}} = \frac{1}{b} \operatorname{covers}^{-1} \frac{bx}{4} + C.$

34. $\int \frac{dx}{\sqrt{ax - x^2}} = \operatorname{vers}^{-1} \frac{2x}{a} + C.$

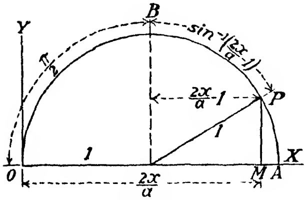
35. $\int \frac{-x\,dx}{\sqrt{ax - x^2}} = \int \frac{a - 2x - a}{2\sqrt{ax - x^2}} dx = \int \frac{(a - 2x)\,dx}{2\sqrt{ax - x^2}} - \frac{a}{2} \int \frac{dx}{\sqrt{ax - x^2}}$

$$= \sqrt{ax - x^2} - \frac{a}{2} \operatorname{vers}^{-1} \frac{2x}{a} + C.$$

36. $\int \frac{dx}{\sqrt{ax - x^2}} = \int \frac{dx}{\sqrt{a^2/4 - (x - a/2)^2}} = \sin^{-1} \left(\frac{2x - a}{a}\right) + C$

$$= \sin^{-1} \left(\frac{2x}{a} - 1\right) + C.$$

Note. — It may be noticed that the result for Example 34 differs from the last result for Example 36. The difference is accounted for by the values of the constants of integration. As may be seen in the figure,



$a=2$, radius $=1$

$$\sin^{-1}\left(\frac{2x}{a}-1\right)+\frac{\pi}{2}=\text{vers}^{-1}\frac{2x}{a},$$

that is,

$$\text{arc } BP+\frac{\pi}{2}=\text{arc } OBP.$$

Each result may be verified by differentiation according to (i) Art. 116. These results illustrate the statement at the end of Art. 117.

122. Reduction Formulas. — A formula by which a differential expression not directly integrable can be reduced to a standard form or a form easier to integrate than the original function, is called a *reduction formula.*

A general reduction formula that has a wide application and is most useful in the reduction of an integral to a known form is the formula for *integration by parts.*

Many special formulas of reduction are obtained by applying this general formula to particular forms.

123. Integration by Parts. — Differentiation gives

$$d(uv) = u\,dv + v\,du.$$

Integrating,

$$uv = \int d(uv) = \int u\,dv + \int v\,du;$$

transposing,

$$\int u\,dv = uv - \int v\,du. \qquad (1)$$

The formula (1) may be used for integrating $u\,dv$ when the integral of $v\,du$ can be found. This method of integration by parts may be adopted when $f(x)\,dx$ is not directly integrable but can be resolved into two factors; one, as dv, directly integrable; and the other, as $v\,du$, a standard form or a form less difficult than $u\,dv$.

No rule can be given for choosing the factors u and dv other than the general direction that the factor of $f(x)\,dx$ taken as dv is first chosen as that part directly integrable, and then what remains whether one or more factors must be taken as u.

When the function given to be integrated contains more than one factor that is directly integrable, there is some choice to be exercised in selecting the factor dv; and in some cases a different choice may be necessary, if the first choice results in $v\,du$ being non-integrable. It may be that one or more applications of the formula to $\int v\,du$ will be effective.

The use of the formula is illustrated in the following examples, the formula being written,

$$\int f(x)\,dx = \int u\,dv$$
$$= uv - \int v\,du. \tag{1}$$

Example 1.— $\int \sin^{-1}\frac{x}{a}\,dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2} + C.$

Let $\quad dv = dx;\quad$ then $\quad u = \sin^{-1}\frac{x}{a},$

$$v = x, \qquad du = \frac{dx}{\sqrt{a^2 - x^2}}.$$

Substituting in (1):

$$\int \sin^{-1}\frac{x}{a}\,dx = x\sin^{-1}\frac{x}{a} - \int \frac{x\,dx}{\sqrt{a^2 - x^2}}$$
$$= x\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2} + C.$$

(Compare example in Art. 118.)

Example 2.— $\int x \cdot \cos x\,dx = x \cdot \sin x - \int \sin x\,dx$

$$= x\sin x + \cos x + C.$$

Example 3. —

$$\int x^2 \sin x\, dx = 2x \sin x - (x^2 - 2) \cos x + C$$

$$\int x^2 \cdot \sin x\, dx = x^2(-\cos x) + 2\int \cos x \cdot x\, dx$$

$$= x^2(-\cos x) + 2(x \sin x + \cos x) + C, \text{ by Ex. 2,}$$

$$= 2x \sin x - (x^2 - 2) \cos x + C.$$

Example 4. —

$$\int x^n \log x\, dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C.$$

$$\int \log x \cdot x^n\, dx = \frac{x^{n+1}}{n+1} \cdot \log x - \int \frac{x^{n+1}}{n+1} \cdot \frac{dx}{x}.$$

$$= \frac{x^{n+1}}{n+1} \log x - \int \frac{x^{n+1}}{(n+1)^2} + C$$

$$= \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C.$$

Example 5. — $\int xe^x\, dx = xe^x - e^x + C.$

$$\int e^x \cdot x\, dx = e^x \cdot \tfrac{1}{2} x^2 - \tfrac{1}{2}\int x^2 \cdot e^x\, dx.$$

The last form is not so simple as the original, indicating that a different choice of factors should be made. Another choice gives

$$\int x \cdot e^x\, dx = x \cdot e^x - \int e^x \cdot dx.$$

$$= xe^x - e^x + C.$$

Example 6. —

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C. \qquad (1)$$

$$\int \sqrt{a^2 - x^2} \cdot dx = \sqrt{a^2 - x^2} \cdot x + \int \frac{x^2\, dx}{\sqrt{a^2 - x^2}}$$

$$= x\sqrt{a^2 - x^2} + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} + C' - \int \sqrt{a^2 - x^2}\, dx.$$

Transposing the last term and dividing by 2 gives (1).

Example 7.* —

$$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2 \log\left(x+\sqrt{x^2+a^2}\right) + C \qquad (2)$$

$$= \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2 \sinh^{-1}\frac{x}{a} + C'. \qquad (2')$$

Example 8.* —

$$\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2 \log\left(x+\sqrt{x^2-a^2}\right) + C \qquad (3)$$

$$= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2 \cosh^{-1}\frac{x}{a} + C'. \qquad (3')$$

Note. — The integrals of the three last examples are of sufficient importance to be considered as *standard forms.*

Example 9. —

$$\int x\left(\frac{e^x - e^{-x}}{2}\right)dx = \int x \cdot \sinh x\,dx = x\cosh x - \sinh x + C.$$

$$\int x \cdot \sinh x\,dx = x \cdot \cosh x - \int \cosh x \cdot dx$$

$$= x\cosh x - \sinh x + C.$$

EXERCISE XXIV.

Verify the following by $\int u\,dv = uv - \int v\,du.$

1. $\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C.$

2. $\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{1}{2}a\log(a^2+x^2) + C.$

3. $\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{1}{2}a\log(a^2+x^2) + C.$

4. $\int \log x\,dx = x(\log x - 1) + C.$

5. $\int x\log x\,dx = \frac{1}{2}x^2(\log x - \frac{1}{2}) + C.$

6. $\int x^3\log x\,dx = \frac{1}{4}x^4(\log x - \frac{1}{4}) + C.$

* By same method as in Example 6.

7. $\int x\,(e^{ax} + e^{-ax})\,dx = \frac{x}{a}(e^{ax} - e^{-ax}) - \frac{1}{a^2}(e^{ax} + e^{-ax}) + C.$

8. $\int e^x \sin x\,dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$

Take $u = e^x$ and apply the formula, then take $u = \sin x$, apply formula; add results.

9. $\int x^2 \cos x\,dx = 2\,x \cos x + (x^2 - 2) \sin x + C.$

10. $\int (\log x)^2\,dx = x\,[(\log x)^2 - 2 \log x + 2] + C.$

11. $\int x^n (\log x)^2\,dx = \frac{x^{n+1}}{n+1}\left[(\log x)^2 - \frac{2}{n+1} \log x + \frac{2}{(n+1)^2}\right] + C$

12. $\int \frac{x^2\,dx}{1+x^2} \tan^{-1} x = \left(x - \frac{1}{2} \tan^{-1} x\right) \tan^{-1} x - \log\left(\sqrt{1+x^2}\right) + C.$

124. Reduction Formulas for Binomials. — By applying the formula for parts to $\int x^m (a + bx^n)^p\,dx$ that integral may be made to depend upon a similar integral, with either m or p numerically diminished.

There are four such formulas, which are useful for reference, but there is no need that they should be memorized.

$$\int x^m (a + bx^n)^p\,dx = \frac{x^{m-n+1}(a + bx^n)^{p+1}}{b\,(np + m + 1)} - \frac{a\,(m - n + 1)}{b\,(np + m + 1)} \int x^{m-n} (a + bx^n)^p\,dx, \quad \text{(A)}$$

or

$$\frac{x^{m+1}(a + bx^n)^p}{np + m + 1} + \frac{anp}{np + m + 1} \int x^m (a + bx^n)^{p-1}\,dx, \quad \text{(B)}$$

or

$$\frac{x^{m+1}(a+bx^n)^{p+1}}{a\,(m+1)} - \frac{b\,(np+m+n+1)}{a\,(m+1)} \int x^{m+n} (a+bx^n)^p\,dx, \quad \text{(C)}$$

or

$$-\frac{x^{m+1}(a+bx^n)^{p+1}}{an\,(p+1)} + \frac{np+m+n+1}{an\,(p+1)} \int x^m (a+bx^n)^{p+1}\,dx. \quad \text{(D)}$$

Formulas (A) and (B) are used when the exponent to be

reduced, m or p, is positive; (A) changing m into $m - n$, and (B) changing p into $p - 1$.

Formulas (C) and (D) are used when the exponent to be reduced, m or p, is negative; (C) changing m into $m + n$, and (D) changing p into $p + 1$.

When any denominator becomes zero the formula is inapplicable, and the integral can be obtained by some method without the use of reduction formulas.

Formulas (A) and (B) fail when $np + m + 1 = 0$.
Formula (C) fails when $m + 1 = 0$.
Formula (D) fails when $p + 1 = 0$.

EXERCISE XXV.

1. $$\int \frac{x^m\,dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C, \text{ when } m = 0, \text{ Standard Form XVIII.}$$

$$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = \int (a^2-x^2)^{-\frac{1}{2}}\,x\,dx = -\sqrt{a^2-x^2}+C, \text{ when } m=1. \quad (1)$$

$$\int \frac{x^m\,dx}{\sqrt{a^2-x^2}} = \int x^m (a^2 - x^2)^{-\frac{1}{2}}\,dx = -\frac{x^{m-1}\sqrt{a^2-x^2}}{m}$$
$$+\frac{(m-1)\,a^2}{m}\int \frac{x^{m-2}\,dx}{\sqrt{a^2-x^2}}, \quad \text{by (A).}$$

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = \int x^2 (a^2 - x^2)^{-\frac{1}{2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2}$$
$$+\frac{a^2}{2}\sin^{-1}\frac{x}{a} + C, \text{ when } m = 2. \quad (2)$$

$$\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = \int x^3 (a^2 - x^2)^{-\frac{1}{2}}\,dx = -\frac{x^2}{3}\sqrt{a^2-x^2}$$
$$+\frac{2}{3}a^2\int \frac{x\,dx}{\sqrt{a^2-x^2}}, \text{ when } m = 3,$$
$$= -\frac{x^2}{3}\sqrt{a^2-x^2} - \frac{2}{3}a^2\sqrt{a^2-x^2} + C, \text{ by (1).} \quad (3)$$

$$\int \frac{x^4\,dx}{\sqrt{a^2-x^2}} = \int x^4 (a^2 - x^2)^{-\frac{1}{2}}\,dx = -\frac{x^3}{4}\sqrt{a^2-x^2}$$
$$+\frac{3}{4}a^2\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}, \text{ when } m = 4,$$
$$= -\left(\frac{x^3}{4}+\frac{3\,a^2x}{4\cdot 2}\right)\sqrt{a^2-x^2}+\frac{3\,a^4}{4\cdot 2}\sin^{-1}\frac{x}{a}+C, \text{ by (2).} \quad (4)$$

2. $\displaystyle\int x^m\sqrt{a^2-x^2}\,dx=\frac{x^{m+1}\sqrt{a^2-x^2}}{m+2}+\frac{a^2}{m+2}\int\frac{x^m\,dx}{\sqrt{a^2-x^2}}$, by (B).

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2-x^2}},\ \text{when } m=0,$$

$$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.\quad\left(\begin{array}{c}\text{Compare Ex.}\\ \text{6, Art. 123.}\end{array}\right)$$

$$\int x^2\sqrt{a^2-x^2}\,dx=\frac{x^3\sqrt{a^2-x^2}}{4}+\frac{a^2}{4}\int\frac{x^2\,dx}{\sqrt{a^2-x^2}},\ \text{when } m=2,$$

$$=\left(\frac{x^3}{4}-\frac{a^2x}{4\cdot 2}\right)\sqrt{a^2-x^2}+\frac{a^4}{4\cdot 2}\sin^{-1}\frac{x}{a}+C,\ \text{by (2) of Ex. 1.}$$

3. $\displaystyle\int\frac{x^2\,dx}{\sqrt{x^2+a^2}}=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)+C$, by (A),

$$=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+C'.\quad\left(\begin{array}{c}\text{Compare Ex.}\\ \text{7, Art. 123.}\end{array}\right)$$

4. $\displaystyle\int\frac{x^2\,dx}{\sqrt{x^2-a^2}}=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)+C$, by (A),

$$=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+C'.\quad\left(\begin{array}{c}\text{Compare Ex. 8,}\\ \text{Art. 123.}\end{array}\right)$$

5. $\displaystyle\int x^2\sqrt{x^2\pm a^2}\,dx=\frac{x^3\sqrt{x^2\pm a^2}}{4}\pm\frac{a^2}{4}\int\frac{x^2\,dx.}{\sqrt{x^2\pm a^2}}$, by (B),

$$=\left(\frac{x^3}{4}\pm\frac{a^2x}{4\cdot 2}\right)\sqrt{x^2\pm a^2}-\frac{a^4}{4\cdot 2}\log\left(x+\sqrt{x^2\pm a^2}\right)+C,\ \text{by Exs. 3, 4.}$$

6. $\displaystyle\int x^m\sqrt{x^2\pm a^2}\,dx=\frac{x^{m+1}\sqrt{x^2\pm a^2}}{m+2}\pm\frac{a^2}{m+2}\int\frac{x^m}{\sqrt{x^2\pm a^2}}$, by (B).

7. $\displaystyle\int\frac{dx}{x^2\sqrt{x^2\pm a^2}}=\mp\frac{\sqrt{x^2\pm a^2}}{a^2x}+C.$

$$\int x^{-2}(x^2\pm a^2)^{-\frac{1}{2}}\,dx=\frac{x^{-1}(x^2\pm a^2)^{\frac{1}{2}}}{\mp a^2}$$

$$-\frac{(-1-2+2+1}{\mp a^2}\int x^0(x^2\pm a^2)^{-\frac{1}{2}}\,dx,\ \text{by (C).}$$

8. $\displaystyle\int\frac{dx}{x^2\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{a^2x}+C.$

$$\int x^{-2}(a^2-x^2)^{-\frac{1}{2}}\,dx=\frac{x^{-1}(a^2-x^2)^{\frac{1}{2}}}{-a^2}$$

$$-\frac{-1(-1-2+2+1)}{-a^2}\int x^0(a^2-x^2)^{-\frac{1}{2}}\,dx,\ \text{by (C).}$$

9. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{x}{a} + C.$

$$\int x^{-3}(x^2-a^2)^{-\frac{1}{2}}dx = \frac{x^{-2}(x^2-a^2)^{\frac{1}{2}}}{2a^2} + \frac{1}{2a^2}\int x^{-1}(x^2-a^2)^{-\frac{1}{2}}dx, \text{ by (C)},$$

$$= \frac{(x^2-a^2)^{\frac{1}{2}}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{x}{a} + C, \text{ by Standard Form XXI.}$$

10. $\displaystyle\int \frac{dx}{x(x^2-a^2)^{\frac{3}{2}}} = -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\sec^{-1}\frac{x}{a} + C.$

$$\int x^{-1}(x^2-a^2)^{-\frac{3}{2}}dx = -\frac{x^0(x^2-a^2)^{-\frac{1}{2}}}{a^2}$$

$$-\frac{1}{a^2}\int x^{-1}(x^2-a^2)^{-\frac{1}{2}}dx, \quad \text{by (D)},$$

$$= -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\sec^{-1}\frac{x}{a} + C.$$

11. $\displaystyle\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C.$

$$\int (a^2-x^2)^{-\frac{3}{2}}dx = -\frac{x(a^2-x^2)^{-\frac{1}{2}}}{2a^2(-\frac{1}{2})}$$

$$+\frac{-3+0+2+1}{2a^2(-\frac{1}{2})}\int x^{\circ}(a^2-x^2)^{-\frac{1}{2}}dx, \quad \text{by (D).}$$

12. $\displaystyle\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + C,$

$$\text{or } \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\text{vers}^{-1}\frac{x}{a} + C'.$$

$$\int \sqrt{2ax-x^2}\,dx = \int x^{\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx. \quad \left(\begin{array}{c}\text{Apply (A) and (B)}\\ \text{in succession.}\end{array}\right)$$

Or

$$\int \sqrt{2ax-x^2}\,dx = \int \sqrt{a^2-(x-a)^2}\,dx$$

$$= \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + C, \text{ by Ex. 2},$$

$$= \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\text{vers}^{-1}\frac{x}{a} + C'. \quad \left(\begin{array}{c}\text{See Ex. 36,}\\ \text{Exercise XXIII.}\end{array}\right)$$

13. $\displaystyle\int x^m\sqrt{2ax-x^2}\,dx = \int x^{m+\frac{1}{2}}\sqrt{2a-x}\,dx = -\frac{x^{m-1}(2ax-x^2)^{\frac{3}{2}}}{m+2}$

$$+\frac{(2m+1)a}{m+2}\int x^{m-1}\sqrt{2ax-x^2}\,dx, \quad \text{by (A).}$$

14. $$\int \frac{x^m\,dx}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\sqrt{2ax-x^2}}{m} + \frac{(2m-1)a}{m}\int \frac{x^{m-1}\,dx}{\sqrt{2ax-x^2}}, \quad \text{by (A).}$$

15. $$\int \frac{dx}{x^m\sqrt{2ax-x^2}} = -\frac{\sqrt{2ax-x^2}}{(2m-1)ax^m} + \frac{m-1}{(2m-1)a}\int \frac{dx}{x^{m-1}\sqrt{2ax-x^2}}, \quad \text{by (C).}$$

16. $$\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{x}{\sqrt{a^2-x^2}+a} + C.$$

$$\int x^{-3}(a^2-x^2)^{-\frac{1}{2}}\,dx = \frac{x^{-3+1}(a^2-x^2)^{\frac{1}{2}}}{a^2(-3+1)} - \frac{-1(-1-3+2+1)}{-2a^2}\int x^{-1}(a^2-x^2)^{-\frac{1}{2}}\,dx, \quad \text{by (C),}$$

$$= -\frac{\sqrt{a^2-x^2}}{2a^2x^2} + \frac{1}{2a^2}\int \frac{dx}{x\sqrt{a^2-x^2}}. \quad \text{See Ex. 18.}$$

17. $$\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} - \frac{1}{2a^3}\log\frac{x}{a+\sqrt{x^2+a^2}} + C.$$

$$\int x^{-3}(x^2+a^2)^{-\frac{1}{2}}\,dx = \frac{x^{-3+1}(x^2+a^2)^{\frac{1}{2}}}{a^2(-3+1)} - \frac{1(-1-3+2+1)}{-2a^2}\int x^{-1}(x^2+a^2)^{-\frac{1}{2}}\,dx, \quad \text{by (C),}$$

$$= -\frac{\sqrt{x^2+a^2}}{2a^2x^2} - \frac{1}{2a^2}\int \frac{dx}{x\sqrt{x^2+a^2}}. \quad \text{See Ex. 19.}$$

18. $$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\log\frac{x}{\sqrt{a^2-x^2}+a} + C.$$

Here $m+1=0$, therefore Formula (C) fails.
Let $a^2-x^2=z^2$; $\therefore\quad -x\,dx = z\,dz,\ x^2=a^2-z^2.$

$$\therefore\quad \int \frac{dx}{x\sqrt{a^2-x^2}} = \int \frac{dz}{z^2-a^2} = \frac{1}{2a}\log\frac{a-z}{a+z} + C$$

$$= \frac{1}{2a}\log\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} + C$$

$$= \frac{1}{a}\log\frac{x}{a+\sqrt{a^2-x^2}} + C.$$

19. $\displaystyle\int\frac{dx}{x\sqrt{x^2+a^2}}=\frac{1}{a}\log\frac{x}{\sqrt{x^2+a^2}+a}+C.$

Again $m+1=0$, and Formula (C) fails.
Let $a^2+x^2=z^2$; $\therefore\quad x\,dx=z\,dz,\ x^2=z^2-a^2.$

$$\therefore\quad \int\frac{dx}{x\sqrt{x^2+a^2}}=\int\frac{dz}{z^2-a^2}=\frac{1}{2a}\log\frac{z-a}{z+a}+C$$

$$=\frac{1}{2a}\log\frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+a^2}+a}+C$$

$$=\frac{1}{a}\log\frac{x}{\sqrt{x^2+a^2}+a}+C.$$

Note. — The integrals of Examples 18 and 19 may be considered as additional *standard forms*.

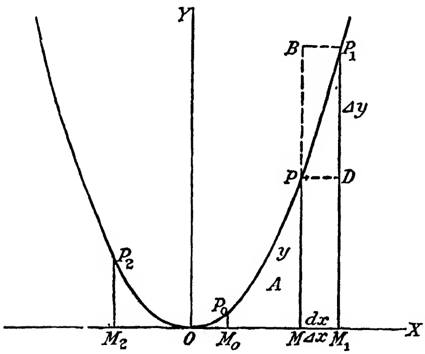
20. $\displaystyle\int\frac{\sqrt{a^2-x^2}}{x}dx=\sqrt{a^2-x^2}+a\log\frac{x}{a+\sqrt{a^2-x^2}}+C.$ (By (B) and Ex. 18.)

CHAPTER II.

DEFINITE INTEGRALS. AREAS.

INTEGRAL CURVES. LENGTH OF CURVES.

125. Geometric Meaning of $\int f(x)\,dx$. — As the representation of an integral by an area between a curve and an axis is of fundamental significance, and as the effort to find an expression for the area of plane figures bounded by curved lines gave rise to the Integral Calculus, such representation will be given further treatment than illustrated in the examples of Art. 115.



Let P_2OP_1 be the locus of $y = f(x)$, and let the area between the curve and the x-axis be conceived as generated by the variable ordinate MP or y, as the point (x, y) moves along the curve and x increases. Let A denote the area bounded by the x-axis, $y = f(x)$, some undetermined fixed ordinate as M_0P_0 or M_2P_2, and the moving ordinate MP.

Let $\Delta x = dx$ be MM_1; then, while ΔA the actual increment of the area A is MPP_1M_1, dA is $MP\,DM_1$, the increment that A *would* get if, at the value M_0P_0PM, the change of A became uniform and so continued while x increased uniformly from the value OM to OM_1. Hence

$$dA = MP\,DM_1 = y\,dx = f(x)\,dx,$$

$$\therefore \quad A = \int y\,dx = \int f(x)\,dx,$$

where A is indeterminate so long as the fixed ordinate M_0P_0 or M_2P_2 is indeterminate.

126. Derivative of an Area. — Since $dA = y\,dx$, $\dfrac{dA}{dx} = y$; that is, *the derivative of the area with respect to x is the ordinate of the bounding curve.*

This important result may be obtained by the method of limits also, taking the increments infinitesimal. Thus,

$$\Delta A = MP\,P_1M_1,$$

$$\Delta A > y\,\Delta x, \quad \text{and} \quad \Delta A < (y + \Delta y)\,\Delta x,$$

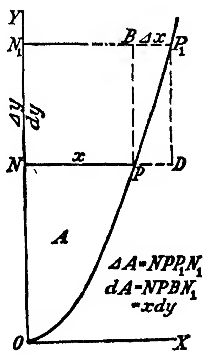
$$\therefore \quad y\,\Delta x < \Delta A < (y + \Delta y)\,\Delta x,$$

$$y < \frac{\Delta A}{\Delta x} < (y + \Delta y),$$

$$\therefore \quad \frac{dA}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta A}{\Delta x} = y,$$

since

$$\lim_{\Delta x \doteq 0} (y + \Delta y) = y, \quad \Delta y \doteq 0 \quad \text{as} \quad \Delta x \doteq 0.$$



In case y decreases as x increases, the curve falls from P to P_1, and the inequality signs are reversed, but the result is the same.

Let A be the area between the y-axis and the curve; then,

$$\frac{dA}{dy} = \lim_{\Delta y \doteq 0} \frac{\Delta A}{\Delta y} = x, \quad \text{or} \quad dA = x\,dy,$$

$$\therefore \quad A = \int x\,dy.$$

Here, *the derivative of the area with respect to y is the abscissa of the bounding curve.*

127. The Area under a Curve. — Let the curve $y = f(x)$ of Art. 125 be $y = x^2$.

Let $OM_0 = a$ and $OM_1 = b$; then

$$M_0P_0PM = A = \int x^2\,dx = \frac{x^3}{3} + C.$$

As the area is measured from $x = a$,

$$\therefore \quad A = 0 = \frac{a^3}{3} + C, \quad C = -\frac{a^3}{3},$$

$$\therefore \quad A_x = \frac{x^3}{3} - \frac{a^3}{3}, \tag{1}$$

where A_x is the variable area $M_0P_0\,PM$. Making $x = b$ in (1) gives

$$M_0P_0P_1M_1 = A_b = \frac{b^3}{3} - \frac{a^3}{3}. \tag{2}$$

The usual notation is

$$A = \int_a^b x^2\,dx = \frac{x^3}{3}\Big]_a^b = \frac{b^3}{3} - \frac{a^3}{3}. \tag{2'}$$

128. Definite Integral. — In general, when $b > a$ the increment produced in the indefinite integral $F(x) + C$ by the increase of x from a to b is

$$F(b) + C - (F(a) + C) \equiv F(b) - F(a).$$

This increment of the indefinite integral of $f(x)\,dx$ is called "the *definite* integral of $f(x)\,dx$ between the limits a and b," and is denoted by $\int_a^b f(x)\,dx$. Hence,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The operation is that of finding the increment of the indefinite integral of $f(x)\,dx$ from $x = a$ to $x = b$, where b is called the *upper* or *superior* limit, and a the *lower* or *inferior* limit, although they are more precisely termed "end values" of

the variable, as they are not "limits" in the usual sense of the word. If the upper end value is variable, then

$$\int_a^x f(x)\,dx = F(x)\Big]_a^x = F(x) - F(a).$$

When the lower end value a is arbitrary, $-F(a)$ may be represented by an arbitrary constant C, hence

$$\int_a^x f(x)\,dx = F(x) + C.$$

Since

$$\int f(x)\,dx = F(x) + C,$$

an indefinite integral is an integral whose upper end value is the variable and whose lower end value is arbitrary.

Hence, when the integral is represented by an area and the area is known for some value a of x,

$$A = \int f(x)\,dx = F(x) - F(a),$$

where C is $-F(a)$, and the area A is determinate.

If the area under the curve $y = x^2$ (Art. 127) be reckoned from $x = 0$; when x is zero, A is zero, therefore, C is zero

and $$A_x = \frac{x^3}{3}, \text{ the area of } OPM. \tag{3}$$

(See Example 4, Art. 115.)

$$A_y = \int y^{\frac{1}{2}}\,dy = \tfrac{2}{3}\,y^{\frac{3}{2}} = \tfrac{2}{3}\,x^3, \text{ the area of } OPN. \tag{3'}$$

Making $x = a$ in (3) gives

$$A_a = \frac{a^3}{3}, \text{ the area of } OP_0M_0, \tag{4}$$

and making $x = b$, gives

$$A_b = \frac{b^3}{3}, \text{ the area of } OP_1M_1. \tag{5}$$

As definite integrals,

$$A = \int_0^a x^2\,dx = \frac{x^3}{3}\Big]_0^a = \frac{a^3}{3}, \quad \text{or} \quad \frac{x^3}{3}\Big]_0^b = \frac{b^3}{3}.$$

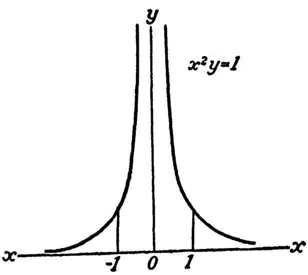
If, in Example 3 of Art. 115, the area A is from $x = a$ to $x = b$; then

$$A = \int_a^b 2\,x\,dx = x^2\Big]_a^b = b^2 - a^2,$$

the area of a trapezoid. It may be noted that, when the integral is "between limits," it is not customary to write the constant, as it will be eliminated.

129. Positive or Negative Areas. — The area under or above a curve $y = f(x)$, from $x = a$ to $x = b$, will be positive or negative according as y is positive or negative from $x = a$ to $x = b$; hence, when the curve crosses the x-axis, the areas are gotten separately, otherwise the result will be the algebraic sum of the areas and may be zero, since the areas above and below the axis may for some curves be equal. For example, the areas for the curve of sines or of cosines as shown in Art. 140 illustrate the principle.

130. Finite or Infinite Areas — "Limits" Infinite. — From the geometrical meaning of an integral it follows that $f(x)\,dx$ has an integral whenever $f(x)$ is *continuous*, hence the end values a and b are in general taken so that $f(x)$ will be finite, continuous, and have the same sign, from $x = a$ to $x = b$.



If $x = b = \infty$, then

$$\int_a^\infty f(x)\,dx = \lim_{b=\infty} \int_{x=a}^{x=b} f(x)\,dx,$$

where the limit *may*, or may *not*, exist. When the limit of the integral is finite the total area is found; but if as b becomes infinite the integral becomes infinite, then no limit exists and the area up to $x = b$ becomes infinite as b becomes infinite.

Example 1.—

$$A = \int_1^\infty \frac{1}{x^2}\,dx = \lim_{b=\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b=\infty}\left[-\frac{1}{x}\right]_1^b = 1.$$

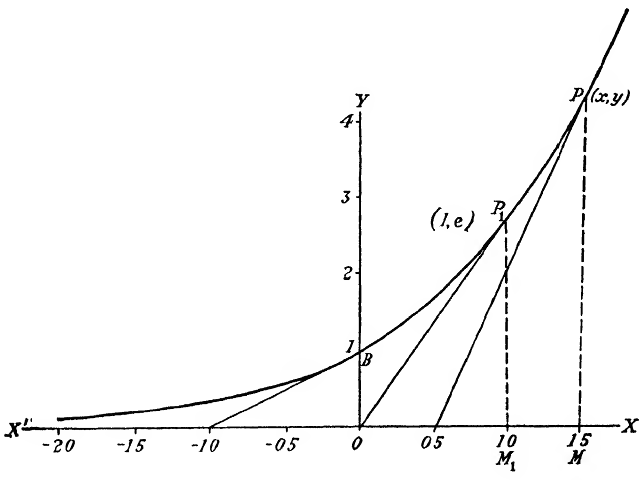
Limit exists although as x approaches zero, $f(x) = \frac{1}{x^2}$ becomes infinite.

$$A = \int_0^\infty \frac{1}{x^2}\,dx = \lim_{b=\infty}\int_0^b \frac{1}{x^2}\,dx = \lim_{b=\infty}\left[-\frac{1}{x}\right]_0^b,$$

which does not exist, since $-\frac{1}{x}\Big]_0^{b=\infty} = \infty$; that is, the area up to $x = b$ becomes infinite as b becomes infinite.

$$A = \int_0^1 \frac{1}{x^2}\,dx = \lim_{a\doteq 0}\int_a^1 \frac{1}{x^2}\,dx = \lim_{a\doteq 0}\left[-\frac{1}{x}\right]_a^1,$$

which does not exist, since $-\frac{1}{x}\Big]_0^1 = \infty$, the area becoming infinite as x approaches zero.



Example 2.—

$$A = \int_{-\infty}^x e^x\,dx = \lim_{a=-\infty}\int_a^x e^x\,dx = \lim_{a=-\infty}\left[e^x\right]_a^x = e^x$$

is total area under curve up to ordinate at $P(x, y)$.

Area to right of y-axis

$$= OMPB = A = \int_0^x e^x\,dx = e^x\Big]_0^x = e^x - 1.$$

Area to left of y-axis

$$= \int_{-\infty}^0 e^x\,dx = \lim_{a=-\infty} \int_a^0 e^x\,dx = \lim_{a=-\infty} \Big[e^x\Big]_a^0 = 1.$$

Note. — When $y = f(x) = e^x$, y, the function, is the ordinate, equals the slope at the end of the ordinate, and may represent the total area under the curve up to the ordinate. (See Art. 138.)

131. Interchange of Limits. — Since the definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a);$$

it follows that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

since the second member is $-[F(a) - F(b)] = F(b) - F(a)$.

It follows also that the *definite* integral is a function of its *limits*, not of its variable; thus

$$\int_a^b f(y)\,dy \text{ has the same value as } \int_a^b f(x)\,dx,$$

each being $F(b) - F(a)$.

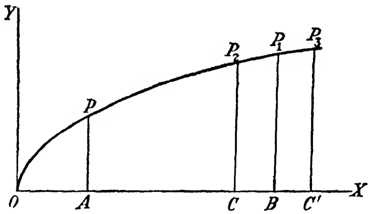
The algebraic sign of a definite integral is changed by an interchange of the limits of integration, and conversely.

132. Separation into Parts. — A definite integral may be separated into parts with other limits or end values. Thus,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \qquad (1)$$

Let the curve $y = f(x)$ be drawn and the ordinates AP_1, BP_1, CP_2, be erected at the points for which $x = a$, $x = b$, $x = c$.

Since area APP_1B = area APP_2C + area CP_2P_1B, (1) follows.



If $OC' = c'$, and $C'P_3$ is the corresponding ordinate, then,

$$\text{area } APP_1B = \text{area } APP_3C' - \text{area } BP_1P_3C';$$

and hence

$$\int_a^b f(x)\,dx = \int_a^{c'} f(x)\,dx - \int_b^{c'} f(x)\,dx$$

$$= \int_a^{c'} f(x)\,dx + \int_{c'}^{b} f(x)\,dx, \text{ by Art. 131.}$$

Note. — It may be seen that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$$

for each is $F(a) - F(o)$. Thus,

$$-\int_0^a f(a-x)\,d(a-x) = -F(a-x)\Big]_0^a = F(a) - F(o)$$

$$= \int_0^a f(x)\,dx.$$

133. Mean Value of a Function. — The mean value of $f(x)$ between the values $f(a)$ and $f(b)$ is

$$\frac{\int_a^b f(x)\,dx}{b-a}.$$

Let area APP_1B represent the definite integral $\int_a^b f(x)\,dx$.

Then

$$\int_a^b f(x)\,dx = \text{area } APP_1B$$

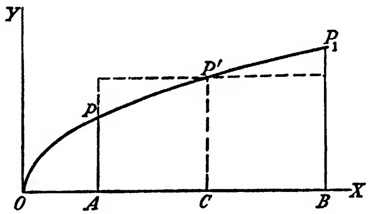
$= \text{area of a rectangle with base } AB \text{ and height greater than } AP \text{ but less than } BP_1$

$= AB \cdot CP'$

$= (b - a) f(c), \text{ where } OC = c.$

Hence $$f(c) = \frac{\int_a^b f(x)\,dx}{b - a},$$

where $f(c)$ is *the mean value of* $f(x)$ for values of x that vary continuously from a to b.



The mean value may be defined to be the height of a rectangle which has a base equal to $b - a$ and an area equivalent to the value of the integral.

Example 1. — To find the mean value of the function $\sqrt{x}$ from $x = 1$ to $x = 4$. Let OPP_1 be the locus of

$$y = \sqrt{x}, \quad OA = 1 \quad \text{and} \quad OB = 4.$$

$$f(c) = \frac{\int_1^4 x^{\frac{1}{2}}\,dx}{4 - 1} = \frac{\left.\frac{2}{3}x^{\frac{3}{2}}\right]_1^4}{3} = \frac{2}{9}[8 - 1] = \frac{14}{9}$$

$$= 1\tfrac{5}{9} = 1.55\tfrac{5}{9} = CP', \text{ mean value.}$$

$$\therefore \quad c = x = (\tfrac{14}{9})^2 = \tfrac{196}{81} = 2.42 = OC.$$

Example 2. — To find the mean value of $\sin\theta$ as θ varies from 0 to $\pi/2$, or from 0 to π.

$$\frac{\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta}{\pi/2 - 0} = \frac{-\cos\theta\Big]_0^{\frac{\pi}{2}}}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi} = 0.6366.$$

$$\frac{\int_0^{\pi} \sin\theta\, d\theta}{\pi - 0} = \frac{-\cos\theta\Big]_0^{\pi}}{\pi} = \frac{2}{\pi} = 0.6366.$$

Example 3. — To find the mean length of the ordinates of a semi-circle of radius a, the ordinates for equidistant intervals on the arc.

$$\frac{\int_0^{\pi} a\sin\theta\, d\theta}{\pi - 0} = \frac{-a\cos\theta\Big]_0^{\pi}}{\pi} = \frac{2a}{\pi} = 0.6366\, a.$$

Example 4. — To find the mean length of the ordinates of a semi-circle of radius a, the ordinates for equidistant intervals along the diameter.

$$\frac{\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx}{a - (-a)} = \frac{\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\Big]_{-a}^{a}}{2a} = \frac{\pi a^2}{2} \times \frac{1}{2a}$$

$$= \frac{\pi}{4}a = 0.7854\, a.$$

134. Evaluation of Definite Integrals.

EXERCISE XXVI.

1. $\int_a^b x^n\, dx = \frac{x^{n+1}}{n+1}\Big]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$

2. $\int_1^2 4x^3\, dx = x^4\Big]_1^2 = 16 - 1 = 15.$

3. $\int_1^x \left(\frac{dx}{x} - \frac{dx}{2-x}\right) = \log x + \log(2-x)\Big]_1^x = \log(2x - x^2)\Big]_1^x$
$= \log(2x - x^2).$

4. $\int_0^{\infty} e^{-ax}\,dx = -\frac{1}{a}e^{-ax}\Big]_0^{\infty} = \frac{1}{a}.$

5. $\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \cos^{-2}\theta \sin\theta\,d\theta = \sec\theta\Big]_0^{\frac{\pi}{4}} = \sqrt{2} - 1.$

6. $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \text{arc}\sin\frac{x}{a}\Big]_0^a = \frac{\pi}{2}.$

7. $\int_0^a \frac{dx}{a^2 + x^2} = \frac{1}{a}\,\text{arc}\tan\frac{x}{a}\Big]_0^a = \frac{\pi}{4a}.$

8. $\int_0^{2r} \frac{2\sqrt{2r}\,dy}{\sqrt{2r - y}} = 8r.$

9. $\int_{-b}^{b} \frac{\pi}{a^4}(y^2 - b^2)^4\,dy = \frac{256\,\pi b^9}{315\,a^4}.$

10. $\int_0^{\infty} \frac{x\,dx}{1 + x^4} = \frac{1}{2}\,\text{arc}\tan x^2\Big]_0^{\infty} = \frac{\pi}{4}.$

11. $\int_0^2 \frac{e^x\,dx}{1 + e^{2x}} = \text{arc}\tan e^x\Big]_0^2 = \text{arc}\tan e^x - \frac{\pi}{4}.$

12. $\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2}\,d\theta = \frac{\pi}{4}.$

13. $\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\,d\theta = \frac{\pi}{4}.$

Note. — Considering the areas between the axes and the graphs:

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx, \text{ where } n \text{ is positive,}$$

$$\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^n x\,dx, \text{ where } n \text{ is positive,}$$

$$\int_0^{\pi} \cos^n x\,dx = 2\int_0^{\frac{\pi}{2}} \cos^n x\,dx, \text{ if } n \text{ is an even integer,}$$

but $= 0$, if n is an odd integer.

135. Areas of Curves. — As has been shown, the formulas in rectangular coördinates are

$$A = \int y\,dx \quad \text{and} \quad A = \int x\,dy.$$

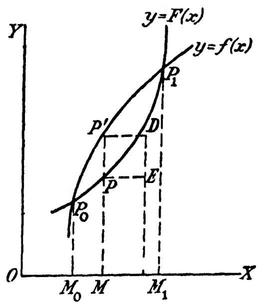
(*a*) Let A denote the area between the curves $y = f(x)$ and $y = F(x)$; let $x = OM$, $dx = PE$; then, the variable area $A = P_0P'P$ and $dA = PP'DE = (f(x) - F(x))\,dx$;

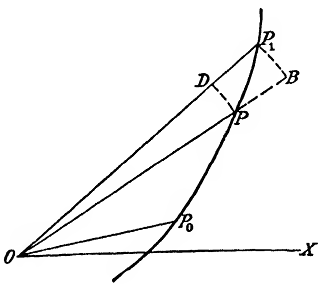
$$\therefore \quad \text{area } P_0P'P_1P = A = \int_{x_0}^{x_1} (f(x) - F(x))\,dx,$$

where the points of intersection are (x_0, y_0) and (x_1, y_1).

If the locus of $y = F(x)$ is the x-axis and x_0 and x_1 are a and b, this formula reduces to

$$A = \int_a^b f(x)\,dx.$$





(*b*) In polar coördinates,

$$A = \tfrac{1}{2}\int \rho^2\,d\theta.$$

For let P_0 be any fixed point (ρ_0, θ_0) and $P(\rho, \theta)$ any variable point.

Consider the area P_0OP to be generated by the radius vector ρ as θ increases from θ_0, and denote it by A. With OP as a radius draw the arc PD and let $d\theta = \angle POP_1$; then

$$dA = OPD = \tfrac{1}{2}\rho \cdot \rho\,d\theta = \tfrac{1}{2}\rho^2\,d\theta,$$

the increment of A, if uniform, as in a circle.

$$\therefore \quad A = \tfrac{1}{2}\int_{\theta_0}^{\theta} \rho^2\,d\theta, \quad \text{or} \quad A = \tfrac{1}{2}\int_0^{\theta} \rho^2\,d\theta, \text{ if } \theta_0 = 0.$$

By method of limits, increments infinitesimal:

$$\tfrac{1}{2}\rho^2\,\Delta\theta < \Delta A < \tfrac{1}{2}(\rho+\Delta\rho)^2\,\Delta\theta, \text{ where } \Delta A = OPP_1,$$

$$\frac{1}{2}\rho^2 < \frac{\Delta A}{\Delta\theta} < \frac{1}{2}(\rho+\Delta\rho)^2,$$

$$\therefore \quad \frac{dA}{d\theta} = \lim_{\Delta\theta \doteq 0}\frac{\Delta A}{\Delta\theta} = \frac{1}{2}\rho^2, \text{ since } \Delta\rho \doteq 0 \text{ as } \Delta\theta \doteq 0.$$

EXERCISE XXVII.

1. Find the area between the parabola $y^2 = 4x$, the x-axis, and the ordinate at any value of x; from $x = 1$ to $x = 4$.

2. Find the area between the parabola $x^2 = 4y$, the y-axis, and the abscissa at any value of y; from $y = 1$ to $y = 9$.

3. Find the area between the two curves $y^2 = 4x$ and $x^2 = 4y$.

4. Find the area between the cubical parabola $4y = x^3$, and the x-axis from $x = 0$ to $x = 2$; from $x = 0$ to $x = -2$; from $x = -2$ to $x = 2$.

5. Find the area of the semi-cubical parabola $4y^2 = x^3$, bounded by the double ordinate at $x = 4$.

6. Find the area between the line $y = x$, and the curve $4y^2 = x^3$, in the first quadrant.

7. Find the area included between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, and the axes of X and Y. *Ans.* $\frac{a^2}{6}$.

8. Find the area bounded by the curve $y(1 + x^2) = x$, and the line $y = \frac{1}{4}x$. *Ans.* $\log 4 - \frac{3}{4}$.

9. Find the area included between the parabola $x^2 = 4ay$, and the witch $y(x^2 + 4a^2) = 8a^3$. *Ans.* $(2\pi - \frac{4}{3})\,a^2$.

10. Find the area bounded by the witch $y(x^2 + 4a^2) = 8a^3$ and its asymptote $y = 0$.

$$\text{Area} = 2\int_0^\infty \frac{8a^3\,dx}{x^2 + 4a^2} = 8a^2 \arctan \frac{x}{2a}\Big]_0^\infty = 4\pi a^2.$$

11. Find the area bounded by the hyperbola $xy = 1$, its asymptote $y = 0$, and the lines $x = 1$ and $x = n$. *Ans.* $\log n$.

When $n = \infty$, $\log n = \infty$; hence the limit does not exist, and the area between the hyperbola and an asymptote is infinite. Since the area is the Napierian logarithm of the superior limit, Napierian logarithms are sometimes called *hyperbolic logarithms*.

12. Find the area of the lemniscate $\rho^2 = a^2 \cos 2\,\theta$.

$$\text{Area} = \frac{4\,a^2}{2} \int_0^{\frac{\pi}{4}} \cos 2\,\theta\, d\theta = a^2.$$

13. Find the area of the cardioid $\rho = 2\,a\,(1 - \cos\theta)$.

$$\text{Area} = 2\,a^2 \int_0^{2\pi} (1 - \cos\theta)^2\, d\theta = 6\,\pi a^2.$$

14. Find the area of the circle $\rho = 2\,a \sin\theta$.

$$\text{Area} = 2\,a^2 \int_0^{\pi} \sin^2\theta\, d\theta = \pi a^2.$$

15. Find the area of the circle $\rho = 2\,a \cos\theta$.

$$\text{Area} = 2\,a^2 \int_0^{\pi} \cos^2\theta\, d\theta = \pi a^2.$$

16. Show that the difference between the areas of the two circles above, from $\theta = 0$ to $\theta = \pi/4$, is a^2; also that the area of one circle intercepted by the other is twice the area of the first circle from $\theta = 0$ to $\theta = \pi/4$.

17. Find the area of the part of the circle

$$\rho = a \sin\theta + b\cos\theta,\ \text{from}\ \theta = 0\ \text{to}\ \theta = \pi/2.$$

Ans. $\dfrac{\pi\,(a^2 + b^2)}{8} + \dfrac{ab}{2}$.

18. Find the area of one loop of the curve $\rho = a \sin 2\,\theta$.

$$\text{Area} = \frac{1}{2} \int_0^{\frac{\pi}{2}} a^2 \sin^2 2\,\theta\, d\theta = \frac{a^2}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 4\,\theta)\, d\theta.$$

Ans. $\dfrac{\pi a^2}{8}$.

19. Find the area between the first and second spire of the spiral of Archimedes $\rho = a\theta$.

$$\text{Area} = \frac{a^2}{2} \int_{2\pi}^{4\pi} \theta^2\, d\theta - \frac{a^2}{2} \int_0^{2\pi} \theta^2\, d\theta = 8\,a^2\pi^3.$$

20. Find the area of one arch of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$.

$$\text{Area} = \int_0^{2\pi a} y\, dx = a^2 \int_0^{2\pi} (1 - \cos\theta)^2\, d\theta,$$

(when $x = 0$, $\theta = 0$; and when $x = 2\,\pi a$, $\theta = 2\,\pi$, and $dx = a\,(1 - \cos\theta)\, d\theta$)

$$= a^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta = 3\,\pi a^2;$$

that is, the area is three times that of the generating circle.

21. Determine the area of the circle $x^2 + y^2 = a^2$.

$$\text{Area} = 4\int_0^a y\,dx = 4\int_0^a \sqrt{a^2 - x^2}\,dx.$$

Using the parametric equations of the circle, $x = a\cos\theta$, $y = a\sin\theta$, where θ is the variable parameter, gives $dx = -a\sin\theta\,d\theta$. Substituting the values of y and dx gives:

$$\text{Area} = 4\int_0^a \sqrt{a^2 - x^2}\,dx = -4\int_{\frac{\pi}{2}}^0 a^2\sin^2\theta\,d\theta,$$

$\left(\text{when } x = a,\ \theta = 0;\quad x = 0, \theta = \dfrac{\pi}{2}\right)$

$$= 4\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\,d\theta, \text{ by Art. 131}$$

$$= 4a^2\int_0^{\frac{\pi}{2}}\left(\frac{1 - \cos 2\theta}{2}\right)d\theta$$

$$= 4a^2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} \quad \left(\begin{matrix}\text{Compare Ex. 14 above}\\ \text{and Ex. 12, Art. 134.}\end{matrix}\right)$$

$$= \pi a^2.$$

To get $\int\sqrt{a^2 - x^2}\,dx$, the indefinite integral; let $x = a\sin\phi$; $dx = a\cos\phi\,d\phi$; then,

$$\int\sqrt{a^2 - x^2}\,dx = a^2\int\cos^2\phi\,d\phi, \text{ where } \phi \text{ is the complement of } \theta \text{ above,}$$

$$= \frac{a^2}{2}\int(1 + \cos 2\phi)\,d\phi$$

$$= \frac{a^2}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right) + C = \frac{a^2}{2}(\phi + \sin\phi\cos\phi) + C$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C,$$

the *indefinite* integral. Compare Ex. 6, Art. 123.

$$\text{Area} = 4\int_0^a\sqrt{a^2 - x^2}\,dx = 4\left[\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C\right]_0^a = \pi a^2, \text{ as above.}$$

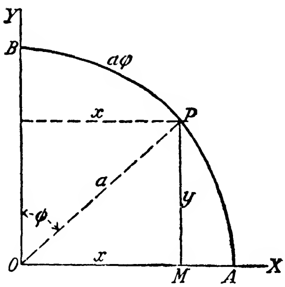
Corollary. — Area of Ellipse $= 4\dfrac{b}{a}\displaystyle\int_0^a\sqrt{a^2 - x^2}\,dx = 4\dfrac{b}{a}\dfrac{\pi a^2}{4} = \pi ab.$

136. To find an Integral from an Area. — An integral may be found from an area, when it can be gotten geometrically from the figure.

Example 1. — Find $\int \sqrt{a^2 - x^2}\, dx$ from the figure of the circle $y = \sqrt{a^2 - x^2}$.

$$\text{Area} = BOMP = BOP + OMP = \tfrac{1}{2} a^2\phi + \tfrac{1}{2} xy$$

$$= \frac{a^2}{2} \sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}.$$



If the initial ordinate is not OB and is undetermined, then,

$$\text{Area} = \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C,$$

as above, C being independent of x and indefinite when the initial ordinate is undetermined.

Example 2. — Find $\int \sqrt{2\,ax - x^2}\, dx$, using circle

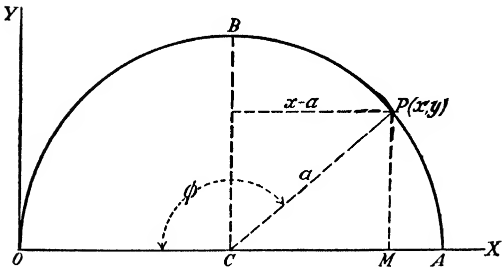
$y = \sqrt{2\,ax - x^2}$.

$$\text{Area} = OBPM = OBPC + PCM$$

$$= \frac{1}{2} a^2\phi + \frac{x-a}{2} y$$

$$= \frac{a^2}{2} \operatorname{vers}^{-1}\frac{x}{a} + \frac{x-a}{2}\sqrt{2\,ax - x^2},$$

or $$\text{Area} = CBPM = BPC + PCM$$
$$= \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + \frac{x-a}{2}\sqrt{2ax - x^2}.$$



If the initial ordinate from which area is reckoned is undetermined, then

$$\int \sqrt{2ax - x^2}\,dx = \frac{x-a}{2}\sqrt{2ax - x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x-a}{a} + C,$$

where $C = \dfrac{\pi a^2}{4}$, if Area $= 0$, when $x = 0$;

or $$\frac{x-a}{2}\sqrt{2ax - x^2} + \frac{1}{2}a^2\,\text{vers}^{-1}\frac{x}{a} + C',$$

where $C' = 0$, if Area $= 0$, when $x = 0$.

As may be seen in the figure,

$$\sin^{-1}\frac{x-a}{a} + \frac{\pi}{2} = OCP = \text{vers}^{-1}\frac{x}{a};$$

that is,

$$\frac{a^2}{2}\left(\sin^{-1}\frac{x-a}{a} + \frac{\pi}{2}\right) = \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + \frac{\pi a^2}{4} = \frac{a^2}{2}\text{vers}^{-1}\frac{x}{a}.$$

(See Note at end of Exercise XXIII.)
Either result gives

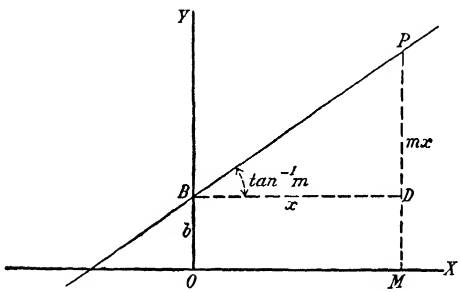
$$\int_0^{2a} \sqrt{2ax - x^2}\,dx = \tfrac{1}{2}\pi a^2 = \text{area of } OBA.$$

Example 3. — Find $\int (mx + b)\, dx$, by means of line $y = mx + b$.

$$\begin{aligned}\text{Area} = OMPB &= BDP + OMDB \\ &= \tfrac{1}{2} x \cdot mx + x \cdot b \\ &= \frac{mx^2}{2} + bx.\end{aligned}$$

If the initial ordinate is not OB, and is undetermined, then

$$\int (mx + b)\, dx = \frac{mx^2}{2} + bx + C.$$



137. Area under Equilateral Hyperbola. — As in the figure of the circle $y = \sqrt{a^2 - x^2}$,

$$\int_0^x \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a}$$

expresses the area $BOMP$ and $a^2 \sin^{-1} \dfrac{x}{a}$ is represented by twice the area of the circular sector BOP; so

$$\int_0^x \sqrt{a^2 + x^2}\, dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \sinh^{-1} \frac{x}{a} \text{ (Ex. 7, Art. 123)}$$

may be shown to express the area $AOMP$ under the equilateral hyperbola $y = \sqrt{a^2 + x^2}$, and $a^2 \sinh^{-1} \dfrac{x}{a}$ to be represented by twice the area of the hyperbolic sector AOP.

To get

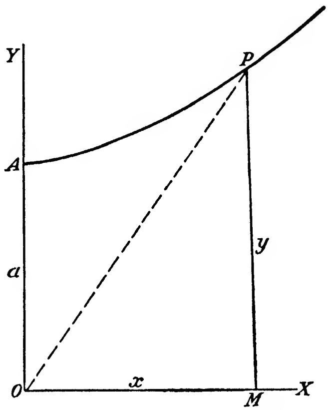
$$\int_0^x \sqrt{a^2+x^2}\,dx; \quad \text{let} \quad x = a\sinh\phi, \ dx = a\cosh\phi\,d\phi;$$

then,

$$\int_0^x \sqrt{a^2+x^2}\,dx = a^2\int \cosh^2\phi\,d\phi = \frac{a^2}{2}(\phi + \sinh\phi\cosh\phi)$$

$$= \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\sinh^{-1}\frac{x}{a},$$

as also in Ex. 7, Art. 123.



If $x = a\cosh\phi$ and $dx = a\sinh\phi\,d\phi$ be substituted in

$$\int \sqrt{x^2-a^2}\,dx;$$

then,

$$\int_0^x \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

(as in Ex. 8, Art. 123), and $\frac{1}{2}a^2\cosh^{-1}\frac{x}{a}$ will be represented by the area of a sector of the equilateral hyperbola

$$y = \sqrt{x^2-a^2}.$$

138. Significance of Area as an Integral. — The units of the number represented by A as the measure of an area will depend upon the units chosen for the abscissa and the ordinate. If the unit for x be one inch and that for y be ten inches, then a unit of A would represent ten square inches; if on the graph the unit of x is one-tenth of an inch and the unit of y is one inch, these representing one and ten inches respectively, an area on the graph of one-tenth of a square inch will represent the area of ten square inches.

The integrals represented by areas may be functions of various kinds, such as lengths, surfaces, volumes, velocities, accelerations, weights, forces, work, etc. Hence, the physical interpretation of the area will depend upon the nature of the quantities represented by abscissa and ordinate.

(*a*) If, in the figure of Art. 125, the ordinate represents velocity and the abscissa represents time, then the area represents distance; and, since velocity

$$v = \frac{ds}{dt} = at,$$

where a is acceleration,

$$\frac{dA}{dt} = v = at.$$

Hence, $$A = \int at\,dt = \tfrac{1}{2}at^2 + C = M_0P_0PM,$$

and since $s = \frac{1}{2}at^2 + s_0$, C is s_0, initial distance or area; and the number of square units of A $(= M_0P_0PM)$ will equal the number of linear units of distance passed over by a moving point in the time $t = M_0M$.

(*b*) If the ordinate represents acceleration and the abscissa still represents time, then the area represents velocity; and since acceleration

$$a = \frac{dv}{dt}, \quad \frac{dA}{dt} = a = \frac{dv}{dt}.$$

Hence,

$$A = \int dv = \int a\,dt = at + C = M_0P_0PM,$$

where a is constant acceleration, and since $v = at + v_0$, C is v_0, initial velocity or area; and the number of square units of A will equal the number of units of velocity acquired by a moving point in the time $t = M_0M$.

(*c*) If the ordinate represents a force acting in a constant direction, and if the abscissa represents the distance through which the force has acted, then the area $A = M_0P_0PM$ will represent the work done by the force acting through the distance represented by M_0M. If the force is constant in magnitude as well as in direction, the area will be a rectangle, since $\dfrac{dA}{ds} = F$, constant, gives $A = \int F\,ds = Fs + C$.

Whether the force is constant or variable the area $A = \int F\,ds$ represents the work done, the area being that under the graph of the equation $y = f(F)$, representing the force. If the force is not constant in direction, the area will still represent the work, provided the ordinate represents the component of the force along the tangent to the path of its point of application.

By means of certain contrivances the curve $y = f(F)$ may be plotted mechanically by the force itself, as, for example, in the steam engine by means of the indicator. Having the curve, the mean force may be easily found; it is given by $\int F\,ds \Big/ M_0M$, the area divided by the distance through which the force acts.

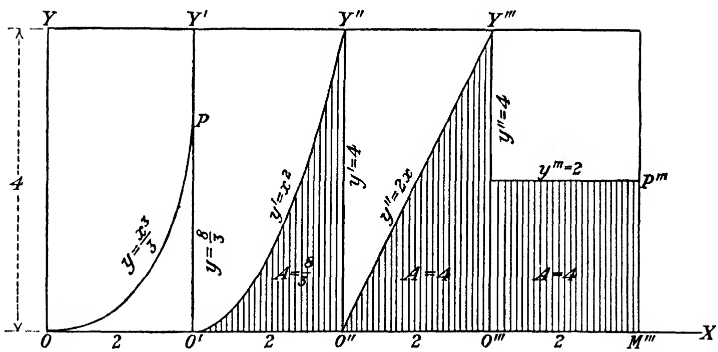
The area may be read off at once by the polar planimeter, and the work done found directly.

It is manifest that a function may be represented graphically either by the *ordinate* of a curve or by the *area* under a curve; if the ordinate is made to represent the function, the

slope of the curve is the derivative of the function; if the area under a curve is taken to represent the function, then the derivative of the function is the ordinate of the curve, since the ordinate is the derivative of the area.

It is usually preferable to represent by the *ordinate* * that which in the investigation is mainly under examination; therefore, if this is the derivative, the latter method, where the area is the function and the ordinate is the derivative, should be used rather than the former method, which is to be used when the function is mainly under consideration. The function e^x is exceptional, in that the ordinate represents the function, the slope of the curve, and the area under the curve. (See Example 2, Art. 130.)

139. Areas under Derived Curves. — It has been shown (Art. 84, figures) by drawing the graphs of a function and its successive derivatives that the variation of the function is exhibited to advantage.



It may now be seen that the area under any derived curve is represented by the ordinate of its primitive curve. Thus the area under the graph of $y = f'(x)$ is represented by the ordinate of $y = f(x)$, that under the graph of $y = f''(x)$ by

* Irving Fisher, *A Brief Introduction to the Infinitesimal Calculus.*

the ordinate of $y = f'(x)$, and so on for the successive derived curves.

Drawing the graphs of $y = f(x) = \frac{x^3}{3}$ and $y = f'''(x) = 2$ together with those of $y = f'(x) = x^2$ and $y = f''(x) = 2x$ (shown in Examples 3 and 4, Art. 115), it is seen that the areas are represented as stated.

It may be seen also that $\int f'(x)\,dx = \int x^2\,dx = \frac{x^3}{3} = A$, being represented by the ordinate of $y = \frac{x^3}{3}$, is an integral function of x^2 and the graph an integral curve of x^2.

If $y = 2$ be the fundamental curve, then $y = 2x$ is the first integral curve; $y = x^2$, the second; $y = \frac{x^3}{3}$, the third; $y = \frac{x^4}{12}$, the fourth; and so on.

140. Integral Curves. — If $F(x)$ has $f(x)$ for its derivative, then $F(x)$ is called an *Integral Function* or simply an *Integral* of $f(x)$. The *General Integral* is $\int f(x)\,dx = F(x) + C$, called also the *Indefinite Integral.*

The graph of an integral function is called an integral curve. If the original or fundamental function is

$$y = f(x), \tag{1}$$

then

$$y = F(x) \tag{2}$$

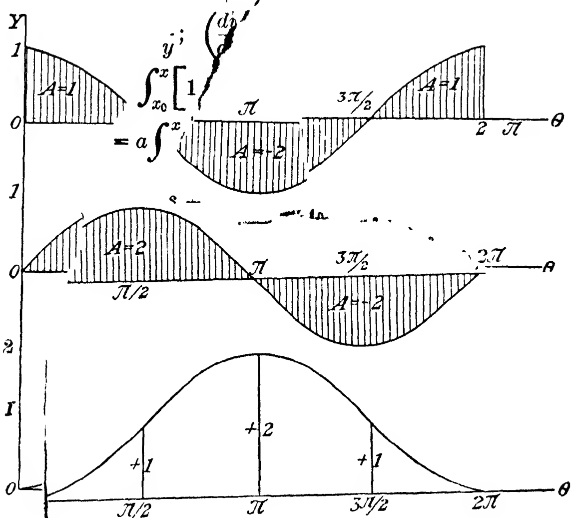
is the *first integral curve* of the curve (1), where $F(x)$ is that integral of $f(x)$ which is zero when x is zero. In the general figure of Art. 125, $F(x)$ is the area OPM under the curve $y = f(x)$; in the figure of Art. 139, if $y = f(x) = x^2$, then $F(x)$ is the area under $y = x^2$ and is the ordinate of the integral curve $y = \frac{x^3}{3}$.

It is manifest that for the same abscissa x, the number that indicates the *length* of the ordinate of the first integral

curve is the same as the number that represents the *area* between the original curve, the axis (or axes for some functions), and the ordinate for this same abscissa. Hence, the ordinates of the first integral curve may represent the areas of the original curve bounded as stated.

It may be seen also that for the same abscissa the number that expresses the slope of the first integral curve is the same as the number that measures the length of the ordinate of the original curve. Hence the ordinates of the original curve may represent the slopes of the first integral curve.

The integral curve of the curve of equation (2) is called the second integral curve of the original curve of equation (1). The integral curve of the second is called the third integral curve of the original curve (1), and so on. Thus for any given curve there is a series of successive integral curves.*



* The statements in the three paragraphs above with some difference of words are given in Murray's *Integral Calculus*, where a fuller treatment will be found in the Appendix.

The function $\cos\theta$ and the first and second integral curves are shown with their graphs.

Let $y = \cos\theta$ be the fundamental function, then

$$y = \int \cos\theta \, d\theta = \sin\theta + C,$$

where C is zero, as y is zero when θ is zero; and

$$y = \int \sin\theta \, d\theta = -\cos\theta + C,$$

where C is one, as y is seen also that [illegible] o when θ is zero. Hence,

$$y = \sin\theta \quad \text{and} \quad y = 1 - \cos\theta = \operatorname{vers}\theta$$

are the first and second integral curves of the curve $y = \cos\theta$.

It is seen that the ordinate of the first integral curve at $\theta = \pi/2$ is $+1$, that number being the same number that measures the area under the fundamental curve for the same abscissa; the ordinate being zero at $\theta = \pi$ indicates that the algebraic sum of the areas of the fundamental curve is zero and hence that the area below from $\theta = \pi/2$ to π is exactly equal to that above from $\theta = 0$ to $\pi/2$; the ordinate being zero again at $\theta = 2\pi$ indicates that the areas of the fundamental curve above and below are exactly equal up to $\theta = 2\pi$.

It is manifest that the ordinates of the second integral curve indicate the corresponding areas for the first integral curve, the number being $+2$ from $\theta = 0$ to π and $2 - 2 = 0$ up to $\theta = 2\pi$.

In the case of this function the series of integral curves can be extended indefinitely without any difficulty.

It is manifest also that the ordinate at any point on the fundamental curve gives the slope at the corresponding point on the first integral curve, the ordinate for the first gives the slope of the second, and so on.

The subject of successive integral curves has useful application to problems in mechanics and engineering.

141. Lengths of Curves. — *Rectangular coördinates.* — It has been shown in Art. 10, (d) that $ds^2 = dx^2 + dy^2$; now let s denote the length of the arc whose ends are the points (x_0, y_0) and (x, y), then $ds = \sqrt{dx^2 + dy^2}$; whence

$$s = \int_{x_0}^{x}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx, \tag{1}$$

or

$$s = \int_{y_0}^{y}\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy, \tag{2}$$

according as ds is expressed in terms of x or of y.

In getting the length of any curve, that formula which gives the simpler expression to integrate is the preferable one to use.

EXERCISE XXVIII.

1. Find s of the circle $x^2 + y^2 = a^2$, and the circumference. Here

$$\frac{dy}{dx} = -\frac{x}{y}; \quad \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2};$$

hence $$s = \int_{x_0}^{x}\left[1 + \frac{x^2}{y^2}\right]^{\frac{1}{2}} dx = \int_{x_0}^{x} \frac{(y^2 + x^2)^{\frac{1}{2}}}{y} dx, \quad \text{using (1),}$$

$$= a\int_{x_0}^{x} \frac{dx}{\sqrt{a^2 - x^2}} = a \sin^{-1}\frac{x}{a}\Big]_{x_0}^{x} = a \sin^{-1}\frac{x}{a} - a \sin^{-1}\frac{x_0}{a}.$$

Circumference, $s = 4 a \sin^{-1}\frac{x}{a}\Big]_0^a = 4\frac{\pi}{2}a = 2\pi a.$

2. Find s of the semi-cubical parabola $ay^2 = x^3$. Here

$$\left(\frac{dy}{dx}\right)^2 = \frac{9x}{4a};$$

$$\therefore \quad s = \int_{x_0}^{x}\left(1 + \frac{9x}{4a}\right)^{\frac{1}{2}} dx, \quad \text{using (1),}$$

$$= \frac{8a}{27}\left[\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} - \left(1 + \frac{9x_0}{4a}\right)^{\frac{3}{2}}\right].$$

From the origin, $s = \frac{8a}{27}\left[\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} - 1\right].$

3. Find s of the cycloid $x = a \text{ arc vers } \frac{y}{a} \mp \sqrt{2ay - y^2}$. Here

$$\left(\frac{dx}{dy}\right)^2 = \frac{y}{2a - y};$$

$$\therefore \quad s = \sqrt{2a}\int_{y_0}^{y}(2a-y)^{-\frac{1}{2}}\,dy, \quad \text{using (2)},$$
$$= 2\sqrt{2a}\,[(2a-y_0)^{\frac{1}{2}} - (2a-y)^{\frac{1}{2}}].$$

Making $y_0 = 0$ and $y = 2a$ and taking twice the result gives $8a$ for the length of one arch. (See Ex. 3, Art. 97.)

Note. — Finding the length of a curve is called *rectifying the curve*, since it is getting a straight line of the same length as the curve. The semi-cubical parabola was rectified by William Neil and by Van Heuraet also, and it is the first curve that was absolutely rectified. The second rectification, that of the cycloid, was by Sir Christopher Wren and by Fermat also, and the third was of the cissoid by Huygens. These rectifications were effected before the development of the Calculus.

4. Find s of the parabola $x^2 = 2py$, (x_0, y_0) being the origin. Here $\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{p^2}$;

$$\therefore \quad s = \int_0^x \left[1 + \frac{x^2}{p^2}\right]^{\frac{1}{2}} dx = \frac{1}{p}\int_0^x (p^2 + x^2)^{\frac{1}{2}}\,dx$$
$$= \frac{1}{p}\left[\frac{x}{2}\sqrt{p^2+x^2} + \frac{p^2}{2}\log\left(x + \sqrt{p^2+x^2}\right)\right]_0^x \quad \text{[Ex. 7, Art. 123.]}$$
$$= \frac{x}{2p}\sqrt{p^2+x^2} + \frac{p}{2}\log\frac{x+\sqrt{p^2+x^2}}{p}.$$

5. Find s of the catenary $y = a\cosh\frac{x}{a} = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. Here

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right); \quad \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{\frac{2x}{a}} - 2 + e^{-\frac{2x}{a}}\right);$$

$$\therefore \quad s = \int_0^x \left[1 + \tfrac{1}{4}\left(e^{\frac{2x}{a}} - 2 + e^{-\frac{2x}{a}}\right)\right]^{\frac{1}{2}} dx$$
$$= \int_0^x \left[\tfrac{1}{4}\left(e^{\frac{2x}{a}} + 2 + e^{-\frac{2x}{a}}\right)\right]^{\frac{1}{2}} dx$$
$$= \int_0^x \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)dx = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

6. Find s of the ellipse $y^2 = (1 - e^2)(a^2 - x^2)$, and the length of the curve, e being the eccentricity. Here

$$\frac{dy}{dx} = \frac{-(1-e^2)^{\frac{1}{2}}x}{\sqrt{a^2-x^2}}; \quad \left(\frac{dy}{dx}\right)^2 = \frac{(1-e^2)\,x^2}{a^2-x^2};$$

$$\therefore \quad s = \int_{x_0}^{x}(a^2 - e^2x^2)^{\frac{1}{2}}\frac{dx}{\sqrt{a^2-x^2}}; \tag{1}$$

hence, the length of the elliptic quadrant s_q is

$$s_q = \int_0^a (a^2 - e^2x^2)^{\frac{1}{2}} \frac{dx}{\sqrt{a^2 - x^2}}. \tag{2}$$

For the integration of (1) and (2), see Ex. 5, Art. 159 and Note.

7. Find s of the circle, and the circumference, using the parametric equations $x = a \cos \theta$, $y = a \sin \theta$. Here

$$\left(\frac{dx}{dy}\right)^2 = \frac{\sin^2 \theta}{\cos^2 \theta}; \quad dy = a \cos \theta \, d\theta;$$

$$\therefore \quad s = \int_y^y \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy = \int_{\theta_0}^{\theta} \left[1 + \frac{\sin^2 \theta}{\cos^2 \theta}\right]^{\frac{1}{2}} a \cos \theta \, d\theta$$

$$= a \int_{\theta_0}^{\theta} d\theta = a\,(\theta - \theta_0)\Big]_0^{2\pi} = 2\pi a.$$

More directly;

$$ds = a\,d\theta; \quad \therefore \quad s = a \int_{\theta_0}^{\theta} d\theta = a\,(\theta - \theta_0)\Big]_0^{2\pi} = 2\pi a.$$

8. Find s of the involute of the circle, and length of the arc between $\theta = 0$ and $\theta = \pi$, using the equations

$$x = a\,(\cos \theta + \theta \sin \theta), \quad y = a\,(\sin \theta - \theta \cos \theta).$$

(See Cor. Ex. 2, Art. 97.) *Ans.* $\frac{1}{2} a\theta^2$; $\frac{1}{2} a\pi^2$.

9. Find s of the cycloid, and the length of one arch, using the parametric equations $x = a\,(\theta - \sin \theta)$, $y = a\,(1 - \cos \theta)$. Here

$$dx = a\,(1 - \cos \theta)\,d\theta; \quad dy = a \sin \theta \, d\theta;$$

$$\therefore \quad s = \int_{x_0}^{x} (dx^2 + dy^2)^{\frac{1}{2}} = a\sqrt{2} \int_{\theta_0}^{\theta} (1 - \cos \theta)^{\frac{1}{2}} d\theta$$

$$= 2a \int_{\theta_0}^{\theta} \sin \frac{\theta}{2} d\theta, \text{ since } \sqrt{1 - \cos \theta} = \sqrt{2} \sin \frac{\theta}{2}$$

$$= -4a \cos \frac{\theta}{2}\Big]_{\theta_0}^{\theta} = 4a\left(\cos \frac{\theta_0}{2} - \cos \frac{\theta}{2}\right)\Big]_0^{2\pi};$$

$\therefore \quad l = 8a$, for one arch.

10. Find s of the parabola $y^2 = 2px$, (x_0, y_0) being the origin, and the length of the arc from the vertex to the end of the latus rectum.

$$\textit{Ans. } s = \frac{y}{2p}\sqrt{p^2 + y^2} + \frac{p}{2} \log \frac{y + \sqrt{p^2 + y^2}}{p};$$

$$l = p/2\,[\sqrt{2} + \log(1 + \sqrt{2})].$$

11. (*a*) Find s of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, and the length of the curve. (*a*) *Ans.* $\frac{3}{2} a^{\frac{1}{3}} (x^{\frac{2}{3}} - x_0^{\frac{2}{3}})$; $6a$.

(*b*) When $x = a \sin^3 \theta$, $y = a \cos^3 \theta$. (*b*) *Ans.* $4\left[\frac{3}{2} a \sin^2 \theta\right]_0^{\frac{\pi}{2}} = 6a$.

142. Lengths of Polar Curves. — It has been shown in Art. 77, (3), that for a polar curve $ds^2 = \rho^2\,d\theta^2 + d\rho^2$. Now let s denote the length of the arc whose ends are (ρ_0, θ_0) and (ρ, θ), then $ds = \sqrt{\rho^2\,d\theta^2 + d\rho^2}$; whence

$$s = \int_{\theta_0}^{\theta} \sqrt{\rho^2\,d\theta^2 + d\rho^2} \quad \text{or} \quad s = \int_{\rho_0}^{\rho} \sqrt{\rho^2\,d\theta^2 + d\rho^2}, \qquad (1)$$

according as ds is expressed in terms of θ or of ρ.

If the formulas of the preceding Article for lengths of curves be transformed to polar coördinates by making $x = \rho\cos\theta$ and $y = \rho\sin\theta$, (1) of this Article results.

EXERCISE XXIX.

1. (*a*) Find s of the circle $\rho = 2a\cos\theta$, and the circumference.

$$s = \int_{\theta_0}^{\theta} \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\,d\theta = 2a\int_{\theta_0}^{\theta} d\theta$$

$$= 2a(\theta - \theta_0)\Big]_0^{\pi} = 2\pi a.$$

(*b*) For the circle $\rho = a$;

$$s = \int_{\theta_0}^{\theta} \rho\,d\theta = a\int_{\theta_0}^{\theta} d\theta = a(\theta - \theta_0)\Big]_0^{2\pi} = 2\pi a.$$

2. Find s of the cardioid $\rho = 2a(1 - \cos\theta)$, and total length.

Here $\qquad d\rho = 2a\sin\theta\,d\theta;$

$$\therefore \quad s = \int_{\theta_0}^{\theta} [4a^2(1-\cos\theta)^2 + 4a^2\sin^2\theta]^{\frac{1}{2}}\,d\theta$$

$$= 2a\int_{\theta_0}^{\theta} \sqrt{2}\,(1-\cos\theta)^{\frac{1}{2}}\,d\theta = 4a\int_{\theta_0} \sin\frac{\theta}{2}\,d\theta$$

$$= 8a\left[\cos\frac{\theta_0}{2} - \cos\frac{\theta}{2}\right]_0^{2\pi} = 16a.$$

3. Find s of the spiral of Archimedes $\rho = a\theta$, and the length of the first spire. Here

$$ds = \sqrt{a^2\theta^2 + a^2}\,d\theta = a\sqrt{1+\theta^2}\,d\theta;$$

$$\therefore \quad s = a\int_{\theta_0}^{\theta}\sqrt{1+\theta^2}\,d\theta = \frac{a}{2}\left[\theta\sqrt{1+\theta^2} + \log\left(\theta + \sqrt{1+\theta^2}\right)\right]_{\theta_0=0}^{\theta=2\pi}$$

$$= a\left[\pi\sqrt{1+4\pi^2} + \tfrac{1}{2}\log\left(2\pi + \sqrt{1+4\pi^2}\right)\right].$$

4. Find s of the logarithmic spiral $\rho = e^{a\theta}$ from ρ_0 to ρ, and the length

from the pole ($\rho = 0$) to $\rho = 1$.

$$a\theta = \log \rho; \quad \therefore \quad a\, d\theta = \frac{d\rho}{\rho} \quad \text{or} \quad \rho\, d\theta = \frac{d\rho}{a};$$

$$\therefore \quad s = \int_{\rho_0}^{\rho} \sqrt{\frac{d\rho^2}{a^2} + d\rho^2} = \frac{\sqrt{1+a^2}}{a} \int_{\rho_0}^{\rho} d\rho$$

$$= \frac{\sqrt{1+a^2}}{a} (\rho - \rho_0)\Big]_0^1 = \frac{\sqrt{1+a^2}}{a}.$$

5. Find s of the conchoid $\rho = a \sec \theta$, and the length of the arc from $\theta = 0$ to $\theta = \pi/4$. Here

$$d\rho = a \sec \theta \tan \theta\, d\theta;$$

$$\therefore \quad s = \int_{\theta_0}^{\theta} \sqrt{a^2 \sec^2 \theta + a^2 \sec^2 \theta \tan^2 \theta}\, d\theta = a \int_{\theta_0}^{\theta} \sec^2 \theta\, d\theta$$

$$= a \tan \theta \Big]_{\theta_0}^{\theta} = a (\tan \theta - \tan \theta_0) \Big]_0^{\frac{\pi}{4}} = a.$$

CHAPTER III.

TAYLOR'S THEOREM — EXPANSION OF FUNCTIONS. INTEGRATION BY SERIES. INDETERMINATE FORMS.

143. Law of the Mean. — The mean value of $f(x)$ between the values $f(a)$ and $f(b)$ is, by Art. 133,

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a},$$

where c is some value between a and b.

If the function of x is $\phi(x) = f'(x)$ and x_1 is some constant value between a and x, then

$$\phi(x_1) = f'(x_1) = \frac{\int_a^x f'(x)\,dx}{x-a} = \frac{f(x) - f(a)}{x-a}$$

or
$$f(x) = f(a) + f'(x_1)(x-a), \tag{1}$$

the *Law of the Mean*, or *Theorem of Mean Value.*

If the curve in the figure be the graph of $y = f'(x)$; then the ordinate at P_1 will be $f'(x_1)$, the mean value of $f'(x)$ between $f'(a)$ at P_a and $f'(x)$ at any value x, the integral being represented by the area under the curve from $x = a$ to $x = x$.

If the curve is $y = f(x)$, it may be seen that there must be at least one point P_1 between the points $(a, f(a))$ and $(x, f(x))$ at which the slope of the tangent is equal to the slope of the secant through those points; that is

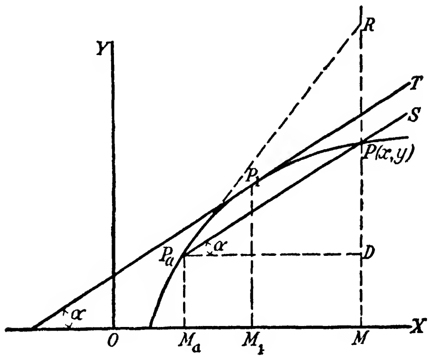
$$f'(x_1) = \frac{f(x) - f(a)}{x-a} = \frac{\Delta y}{\Delta x},$$

and hence (1).

This may be put in the form,

$$\Delta y = \left[\frac{dy}{dx}\right]_{x=x_1} \cdot \Delta x,$$

which may be used to determine increments approximately, and is the *Theorem of Finite Differences.*



The theorem may be extended so as to express in terms of the second derivative the difference made in using the first derivative at $x = a$ in place of its value at $x = x_1$. Thus, if the function of x is $\phi(x) = f''(x)$, and x_2 is some constant value between a and x, then,

$$f''(x_2) = \frac{\int_a^x f''(x)\,dx}{x - a} = \frac{f'(x) - f'(a)}{x - a}, \quad \left(\begin{array}{c}\text{by mean value,}\\ \text{Art. 133,}\end{array}\right)$$

or
$$f'(x) = f'(a) + f''(x_2)(x - a).$$

Integrating this equation between the limits $x = a$ and $x = x$, $f'(a)$ and $f''(x_2)$ being constants, gives

$$f(x) = f(a) + f'(a)(x - a) + f''(x_2)\frac{(x - a)^2}{2}, \qquad (2)$$

a second *Theorem of Mean Value,* or the *Law of the Mean.*

If the tangent at P_a meet the ordinate MP produced at R, then, $MR = f(a) + f'(a)(x - a)$; $MP = f(x)$, and, therefore, both in sign and in magnitude,

$$RP = MP - MR = f''(x_2)\frac{(x-a)^2}{2}.$$

Here the deviation of the curve at P is below the tangent at P_a, $f''(x_2)$ being negative, and, measured along the line of the ordinate MP, is equal to $f''(x_2)\frac{(x-a)^2}{2}$. When the curve is above the tangent, $f''(x_2)$ will be positive and RP upward.

144. Other Forms of the Law of the Mean. — The theorems (1) and (2) may be given in the following forms.

In the theorems x, the symbol for the argument in general has been used for *any* value of the argument, a definite value but not constant. Now, if x_1 be any number between a and x, then $x_1 - a$ and $x - a$ are of the same sign whether a is less or greater than x; therefore, $(x_1 - a)/(x - a)$ is a positive proper fraction, θ say, and $x_1 = a + \theta(x - a)$ will denote any number between a and x.

Letting $x = a + h$, $x - a = h$; then theorem (1) will become

$$f(a+h) = f(a) + hf'(a + \theta h), \tag{1$_a$}$$

and theorem (2) will become

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a + \theta_1 h). \tag{2$_a$}$$

The θ_1 of (2_a) is not necessarily the same as the θ of (1_a). If a is replaced by x, the forms become

$$f(x+h) = f(x) + hf'(x + \theta h), \tag{1$_b$}$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x + \theta_1 h). \tag{2$_b$}$$

If a is made zero and then x is put for h, the forms are

$$f(x) = f(0) + xf'(\theta x), \tag{1_c}$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(\theta_1 x). \tag{2_c}$$

Example 1. — To find the value of θ, if $f(x) = x^2$. Here

$$f'(x) = 2x; \quad f'(a + \theta h) = 2(a + \theta h);$$

and $\quad (a + h)^2 = a^2 + 2ah + h^2 = a^2 + h \cdot 2(a + \frac{1}{2}h);$

also, by (1_a),

$$(a + h)^2 = f(a) + hf'(a + \theta h) = a^2 + h \cdot 2(a + \theta h);$$

hence in this case $\theta = \frac{1}{2}$.

To find at what point on the parabola $y = x^2$, the tangent is parallel to the secant through the points where $x = 1$ and $x = 3$. By theorem (1),

$$f'(x_1) = 2x_1 = \frac{f(x) - f(a)}{x - a} = \frac{9 - 1}{3 - 1} = 4;$$

$\therefore \quad x_1 = 2$; or by theorem (1_a), $x_1 = a + \theta h = 1 + \frac{1}{2}(2) = 2$, since $\theta = \frac{1}{2}$.

Example 2. — To find at what point on the curve $y = \sin x$, the tangent is parallel to the secant through the points where $x = 30°$ and $x = 31°$. Here

$$f'(x_1) = \cos x_1 = \frac{\sin 31° - \sin 30°}{31° - 30°} = \frac{0.51504 - 0.5}{0.01745} = 0.86177;$$

$$\therefore \quad x_1 = \cos^{-1} 0.86177 = 30°\,29';$$

hence $\qquad y_1 = \sin 30°\,29' = 0.50729, \quad \theta = \frac{29}{60}.$

Example 3. — To show that $\sin x$ is less than x but is greater than $x - \frac{1}{2}x^2$.

$$f(x) = \sin x; \quad f'(x) = \cos x; \quad f''(x) = -\sin x;$$

$$f(0) = 0; \quad f'(0) = 1; \quad f''(\theta_1 x) = -\sin(\theta_1 x).$$

By theorem (1_c),

$$\sin x = 0 + x\cos(\theta x), \; < x, \text{ since } \cos(\theta x) < 1.$$

By theorem (2_c),

$$\sin x = 0 + x - \frac{x^2}{2}\sin(\theta_1 x) > x - \frac{x^2}{2}, \text{ since } |\sin(\theta_1 x)| < 1.$$

Example 4. — To show that $\cos x$ is greater than $1 - \frac{1}{2}x^2$.

$$f(x) = \cos x; \quad f'(x) = -\sin x; \quad f''(x) = -\cos x;$$
$$f(0) = 1; \quad f'(0) = 0; \quad f''(\theta_1 x) = -\cos(\theta_1 x).$$

By theorem (2_c),

$$\cos x = 1 - \tfrac{1}{2}x^2\cos(\theta_1 x) > 1 - \tfrac{1}{2}x^2, \text{ since } |\cos(\theta_1 x)| < 1.$$

145. Extended Law of the Mean. — The law of the mean or the theorem of mean value in its several forms may be used to obtain approximate expressions for a given function in the neighborhood of a given point $x = a$. Still closer approximations may be obtained from the law when extended in the form of a series arranged according to ascending powers of $x - a$ with the successive derivatives as constant coefficients.

For values of x near to a, the higher powers of $x - a$ may then become negligible. The most convenient theorem for this purpose is the one which follows.

• **146. Taylor's Theorem.** — *If $f(x)$ is continuous, and has derivatives through the nth, in the neighborhood of a given point $x = a$, then, for any value of x in this neighborhood,*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$
$$\cdots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^n(X)}{n!}(x-a)^n, \quad (1)$$

where X is some unknown quantity between a and x.

The last term

$$R_n(x) = \frac{f^n(X)}{n!}(x-a)^n$$

is the *error* in stopping the series with the nth term, the term in $(x-a)^{n-1}$; and the formula is of practical use only when

this error becomes smaller and smaller as the number of terms is increased.

The form of the *remainder* $R_n(x)$ is seen to differ from the general term of the series only in that the derivative in the coefficient of the power of $(x - a)$ is taken for $x = X$ instead of for $x = a$.

The simplest proof of this theorem is the extension by integration of the law of the mean — a further extension than already used for the theorems of mean value.

Thus, if the function of x is $\phi(x) = f'''(x)$, and X is some unknown constant value between a and x, then,

$$\int_a^x f'''(x)\,dx = f'''(X)(x - a) \quad \text{(by mean value, Art. 133)}$$
$$= f''(x) - f''(a) = f'''(X)(x - a).$$

Integrating this equation between the limits $x = a$ and $x = x$,

$$f'(x) - f'(a) - f''(a)(x - a) = f'''(X)\frac{(x - a)^2}{2},$$

$f''(a)$ and $f'''(X)$ being constants.

Integrating again, $f'(a)$ also being constant, gives

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2} + f'''(X)\frac{(x - a)^3}{2\cdot 3}.$$

As this process can be continued to include the nth derivative, by induction Taylor's Theorem results.

147. Another Form of Taylor's Theorem. — If in the form (1) x is put for a and $(x + h)$ for x, it becomes

$$f(x + h) = f(x) + f'(x)\frac{h}{1} + f''(x)\frac{h^2}{2!} + \cdots$$
$$+ f^{n-1}(x)\frac{h^{n-1}}{(n - 1)!} + f^n(x + \theta h)\frac{h^n}{n!}, \qquad (2)$$

where the last term is the remainder after n terms, and θ is some positive proper fraction. In (1), $(a + \theta(x - a))$ may be used in place of X; and in (2), it becomes $(x + \theta h)$

Another form of the remainder called Cauchy's is

$$R_n(x) = f^n(a + \theta(x - a)) \frac{(x - a)^n (1 - \theta)^{n-1}}{(n - 1)!}.$$

Note. — Taylor's Theorem is the discovery of Dr. Brook Taylor, and was first published by him in 1715. He gave it as a corollary to a theorem in *Finite Differences* and there was no reference to a remainder. It was Lagrange who, in 1772, called attention to its great value and found for the remainder the expression, $\frac{f[a + \theta(x - a)]}{n!}(x - a)^n$, since called by his name. It has become to be regarded as the most important formula in the Calculus.

The formula known as Maclaurin's Theorem, after Colin Maclaurin, was published by him in 1742, but he recognized it as a special case of Taylor's Theorem. The two theorems are virtually identical as either can be deduced from the other.

148. Maclaurin's Theorem. — If, in the form (1), a is made 0;

$$f(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \frac{f^n(X)}{n!}x^n, \qquad (3)$$

where $X = \theta x$ is some unknown quantity between 0 and x.

Another form of Maclaurin's Theorem is

$$f(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{n-1}(0)\,x^{n-1}}{(n-1)!} + \frac{f^n(\theta x)\,x_n}{n!}, \qquad (4)$$

where the last term is the remainder after n terms, and θ is some positive proper fraction.

Cauchy's form of the remainder is

$$R_n(x) = f^n(\theta x)\frac{x^n(1 - \theta)^{n-1}}{(n - 1)!}.$$

149. Expansion of Functions in Series. — It has been shown in examples on Infinite Series how useful it is to be able to represent a function by means of a series. Apart from the purpose of computation such representation may be an aid to an understanding of the properties of functions. Taylor's Theorem and Maclaurin's Theorem furnish a general method of expanding or developing any one of a numerous class of functions into a power series.

For when the error term in Taylor's and in Maclaurin's Theorem approaches zero as n increases, each becomes a convergent infinite series, called Taylor's series and Maclaurin's series for the given function, about the given point $x = a$.

Some functions may be expanded by division, some by the binomial theorem, others by the logarithmic or the exponential series. All of these series are but particular cases of Taylor's Theorem.

150. Another Method of Deriving Taylor's and Maclaurin's Series. — 1. *Maclaurin's Series.* — If a function of a single variable is expanded or developed into a series of terms arranged according to the ascending integral powers of that variable, and the constant coefficients found, the development will be the form of Maclaurin's Theorem without the remainder. Thus, let $f(x)$ and its successive derivatives be continuous in the neighborhood of $x = 0$, say from $x = -a$ to $x = a$, and assume that for values of x within that interval,

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots \tag{1}$$

If equation (1) is identically true, then the equation resulting from differentiating both its members, viz.,

$$f'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots,$$

also is identically true for values of x in some interval that includes zero. For similar values of x, the following equa-

tions resulting from successive differentiations are identically true:

$$f''(x) = 2C + 2 \cdot 3 Dx + 3 \cdot 4 Ex^2 + \cdots.$$
$$f'''(x) = 2 \cdot 3 D + 2 \cdot 3 \cdot 4 Ex + \cdots.$$
$$f^{\text{IV}}(x) = 2 \cdot 3 \cdot 4 E + \cdots.$$
$$\cdots\cdots\cdots\cdots\cdots$$

Putting $x = 0$ in each of these equations gives:

$$A = f(0),\ B = f'(0),\ C = \frac{f''(0)}{2!},\ D = \frac{f'''(0)}{3!},\ E = \frac{f^{\text{IV}}(0)}{4!},\ \text{etc.}$$

Hence, on substituting these values in equation (1),

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)\,x^3}{3!} + \cdots + \frac{f^n(0)\,x^n}{n!} + \cdots, \tag{2}$$

which is Maclaurin's series as in (3), Art. 148, without the remainder.

If $f(x)$ is not continuous, no development according to powers of x is possible. Thus if $f(x) = \log x$, $f(0) = -\infty$.

A power series represents a continuous function, hence no power series in x can be expected to develop $\log x$.

It is evident that, whenever the function or any one of its derivatives is discontinuous for $x = 0$, the function cannot be developed in a Maclaurin's series.

2. *Taylor's Series.* — Let $f(x)$ and its successive derivatives be continuous in the neighborhood of $x = a$, say from $x = a - h$ to $x = a + h$, and assume that for values of x near $x = a$,

$$f(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + E(x-a)^4 + \ldots, \tag{3}$$

is an identically true equation.

Then the following equations resulting from successive differentiations are identically true for values of x near $x = a$.

$$f'(x) = B + 2C(x-a) + 3D(x-a)^2 + 4E(x-a)^3 + \cdots.$$
$$f''(x) = 2C + 2\cdot 3D(x-a) + 3\cdot 4E(x-a)^2 + \cdots.$$
$$f'''(x) = +2\cdot 3D + 2\cdot 3\cdot 4E(x-a) + \cdots.$$
$$f^{\text{IV}}(x) = +2\cdot 3\cdot 4E + \cdots.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Putting $x = a$ in each of these equations gives

$$A = f(a),\ B = f'(a),\ C = \frac{f''(a)}{2!},\ D = \frac{f'''(a)}{3!},\ E = \frac{f^{\text{IV}}(a)}{4!},\ \text{etc.}$$

Hence, on substituting these values in equation (3),

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$
$$+ \frac{f^n(a)}{n!}(x-a)^n + \cdots, \tag{4}$$

which is Taylor's series as in (1), Art. 146, without the remainder.

If in (4) x is put for a and $(x+h)$ for x, it becomes

$$f(x+h) = f(x) + f'(x)\frac{h}{1!} + f''(x)\frac{h^2}{2!} + \cdots$$
$$+ \frac{f^{n-1}(x)}{(n-1)!}h^{n-1} + \cdots, \tag{5}$$

which is Taylor's series as in (2), Art. 147, without the remainder.

Here the development is not according to powers of x, but of some value $(x-a)$ or h near to the value $x = a$. Hence, when the values of the function and all its derivatives are known or can be found for some one value of x, say a, then the value of the function for $x = a + h$ can be found from the development. Thus, when $f(x) = \log x$; $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2!$, $\ldots$ $f^{(n)}(1) = (-1)^{n+1}(n-1)!$; and the series will be

$$\log(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \cdots,$$

which agrees with (1), Ex. 1, Art. 159.

151. Expansion by Maclaurin's and Taylor's Theorems. — If, on applying to a given function any one of these formulas, the last term becomes 0, or approaches 0 as a limit when n becomes infinite, the formula develops this function; if not, the formula fails for this function. That is, if $R_n(x) \doteq 0$ when $n = \infty$, Maclaurin's or Taylor's series is the development of $f(x)$ or $f(x+h)$, respectively.

If $f^n(x)$ increases (or decreases) from $f^n(0)$ to $f^n(x)$, and the sum of the first n terms in Maclaurin's series is taken as the value of $f(x)$, the *error*, being $f^n(\theta x)\frac{x^n}{n!}$, lies between

$$f^n(0)\frac{x^n}{n!} \quad \text{and} \quad f^n(x)\frac{x^n}{n!}.$$

If $f^n(x)$ increases (or decreases) from $f^n(x)$ to $f^n(x+h)$, and the sum of the first n terms in Taylor's series is taken as the value of $f(x+h)$, the *error*, being $f^n(x+\theta h)\frac{h^n}{n!}$, lies between

$$f^n(x)\frac{h^n}{n!} \quad \text{and} \quad f^n(x+h)\frac{h^n}{n!}.$$

152. Since $\frac{x^n}{n!} = \frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdots\frac{x}{n-2}\cdot\frac{x}{n-1}\cdot\frac{x}{n}$,

$$\frac{x^n}{n!} \doteq 0 \textit{ when } x \textit{ is finite and } n = \infty,$$

for last factor approaches zero.

Hence, $R_{n=\infty}(x) \doteq 0$ when $f^n(\theta x)$ is finite, or when $f^n(x+\theta h)$ is finite, in Maclaurin's or Taylor's Theorems, respectively.

153. If $f(x) = f(-x)$, the expansion of $f(x)$ will contain only even powers of x; while if $f(x) = -f(-x)$ the expansion of $f(x)$ will involve only odd powers of x. For examples, see the expansions of $\sin x$ and of $\cos x$ following.

154. Examples. — 1. **Sin x.** — Expansion by Maclaurin's Theorem:

$$\begin{aligned} f(x) &= \sin x, & f(0) &= \sin 0 = 0, \\ f'(x) &= \cos x, & f'(0) &= 1, \\ f''(x) &= -\sin x, & f''(0) &= 0, \\ f'''(x) &= -\cos x, & f'''(0) &= -1, \\ f^{\mathrm{IV}}(x) &= \sin x, & f^{\mathrm{IV}}(0) &= 0, \\ \cdots & & \cdots & \\ f^{n}(x) &= \sin\left(x + \frac{n\pi}{2}\right), & f^{n}(0) &= \sin\left(\frac{n\pi}{2}\right), \\ f^{n}(\theta x) &= \sin\left(\theta x + \frac{n\pi}{2}\right). \end{aligned}$$

Since $\sin\left(\frac{n\pi}{2}\right)$ is 0 or ± 1 according as n is even or odd, the coefficients of the even powers of x will be zero, and only odd powers of x will occur, the terms being alternately positive and negative. Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^n}{n!}\sin\left(\theta x + \frac{n\pi}{2}\right).$$

Here, $$R_n(x) = \frac{x^n}{n!}\sin\left(\theta x + \frac{n\pi}{2}\right),$$

not numerically greater than $\frac{x^n}{n!}$, which has zero for limit;

$\therefore \quad \underset{n=\infty}{R(x)} \doteq 0.$ Hence the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \tag{1}$$

is absolutely convergent for every finite value of x.

The series converges rapidly and may be used for computing the natural sine of any angle expressed in radians.

Thus, for the sin ($5° 43' 46''.5 = \frac{1}{10}$ radian),

$$\sin(0.1 \text{ radian}) = 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \cdots = 0.09983 \ldots.$$

For the $\sin\left(1^\circ = \frac{\pi}{180} \text{ radian} = 0.017453 \ldots\right)$,

$$\sin 1^\circ = \sin\left(\frac{\pi}{180}\right) = \frac{\pi}{180} - \left(\frac{\pi}{180}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{180}\right)^5 \frac{1}{5!} - \cdots = 0.017452 \ldots ;$$

$\therefore$ $\sin 1^\circ = \text{arc } 1^\circ$, to five places of decimals. (See Art. 40.)

2. **Cos x.** — Expansion may be made by Maclaurin's Theorem as is done for the $\sin x$, or the differentiation of the sine series term by term gives the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots , \tag{2}$$

which is absolutely convergent for every finite value of x.

For the cos ($5^\circ 43' 46''.5 = \frac{1}{10}$ radian),

$$\cos (0.1 \text{ radian}) = 1 - \frac{(0.1)^2}{2!} + \frac{(0.1)^4}{4!} - \cdots = 0.995007 \ldots .$$

For the $\cos\left(1^\circ = \frac{\pi}{180} \text{ radian} = 0.017453 \ldots\right)$,

$$\cos 1^\circ = \cos\left(\frac{\pi}{180}\right) = 1 - \left(\frac{\pi}{180}\right)^2 \frac{1}{2!} + \left(\frac{\pi}{180}\right)^4 \frac{1}{4!} - \cdots$$
$$= 0.999847 \ldots ;$$

$\therefore$ $\cos 1^\circ = 1 - 0.00015 \ldots$

3. **Sin $(x + h)$.** — Expansion by Taylor's Theorem. Here $f(x + h) = \sin (x + h)$;

$\therefore$ $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, etc.

Hence,

$$\sin (x+h) = \sin x + h \cos x - \frac{h^2}{2!} \sin x - \frac{h^3}{3!} \cos x + \frac{h^4}{4!} \sin x + \cdots$$
$$= \sin x \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \cdots\right)$$
$$+ \cos x \left(h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!} + \cdots\right) \tag{1}$$
$$= \sin x \cos h + \cos x \sin h, \quad (h \text{ for } x \text{ in Exs. 1 and 2})$$

the well-known relation true for all values of x and h.

When $x = 0$ in (1), $\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!} + \cdots$, as in Ex. 1 for x. The series (1) is rapidly convergent for small values of h. Thus, let $x = \frac{\pi}{6}$ and $h = \frac{1}{100}$ of a radian $= 34' \, 22''.65$; then,

$$\sin\left(\frac{\pi}{6} + \frac{1}{100}\right)^r = \sin\frac{\pi}{6}\left(1 - \frac{(0.01)^2}{2!} + \frac{(0.01)^4}{4!} - \cdots\right)$$
$$+ \cos\frac{\pi}{6}\left(0.01 - \frac{(0.01)^3}{3!} + \frac{(0.01)^5}{5!} - \cdots\right);$$

$$\sin 30° \, 34' \, 22''.65 = 0.5\,(0.99995 \ldots) + 0.86603 \ldots (0.00999 \ldots) = 0.50863 \ldots = 0.5 + 0.00863 \ldots$$

4. **Cos $(x + h)$.** — Expansion may be made in the same way as for $\sin(x + h)$, or the differentiation of the series (1), h being constant, gives

$$\cos(x + h) = \cos x\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \cdots\right)$$
$$- \sin x\left(h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!} + \cdots\right) \tag{2}$$
$$= \cos x \cos h - \sin x \sin h, \text{ the well-known relation.}$$

When $x = 0$ in (2), $\cos h = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \cdots$, as in Ex. 2 for x.

When $x = \frac{\pi}{3}$ and $h = \frac{1}{100}$ of a radian $= 34' \, 22''.65$, in (2); then $\cos 60° \, 34' \, 22''.65 = 0.5\,(0.99995 \ldots) - 0.86603 \ldots (0.00999 \ldots) = 0.49132 \ldots = 0.5 - 0.00868 \ldots$.

5. **a^x and e^x.**—Expansion by Maclaurin's Theorem. Here

$$\begin{aligned} f(x) &= a^x, & f(0) &= a^0 = 1, \\ f'(x) &= a^x \log a, & f'(0) &= \log a, \\ f''(x) &= a^x (\log a)^2, & f''(0) &= (\log a)^2, \\ f'''(x) &= a^x (\log a)^3, & f'''(0) &= (\log a)^3, \\ &\cdots & &\cdots \\ f^n(x) &= a^x (\log a)^n, & f^n(0) &= (\log a)^n, \\ f^n(\theta x) &= a^{\theta x} (\log a)^n. \end{aligned}$$

$$\therefore \quad a^x = 1 + \frac{x \log a}{1} + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \cdots$$
$$+ \frac{(x \log a)^n}{n!} a^{\theta x}.$$

Since $f^n(x) = a^x (\log a)^n$, when a is positive, $f(x)$ and all its successive derivatives are continuous for all values of x.

$$R_n(x) = f_n(\theta x) \frac{x^n}{n!} = \frac{(x \log a)^n}{n!} a^{\theta x}.$$

When x is finite, $a^{\theta x}$ is finite. By Art. 152, $\frac{(x \log a)^n}{n!} \doteq 0$, when $n = \infty$ and $x \log a$ is finite. Hence $R_n(x) \doteq 0$ when $n = \infty$ and x is finite. Therefore, the *exponential series*

$$a^x = 1 + \frac{x \log a}{1} + \frac{(x \log a)^2}{2!} + \cdots + \frac{(x \log a)^{n-1}}{(n-1)!} + \cdots \qquad (1)$$

is the development of a^x when a is positive and x is finite, being then absolutely convergent.

Value of e^x. — Putting $a = e$ in (1), gives (since $\log e = 1$)

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots. \qquad (2)$$

Value of e. — Putting $x = 1$ in (2), gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{(n-1)!} + \cdots$$
$$= 2.718281 \ldots \quad \text{(See Art. 34.)}$$

6. **Log$_a$(1 + x).** — Expansion by Maclaurin's Theorem. Here

$$f(x) = \log_a(1+x), \qquad f(0) = \log_a 1 = 0,$$
$$f'(x) = \frac{m}{1+x}, \qquad f'(0) = m,$$
$$f''(x) = -\frac{m}{(1+x)^2}, \qquad f''(0) = -m,$$
$$f'''(x) = \frac{2m}{(1+x)^3}, \qquad f'''(0) = 2m,$$
$$\cdots\cdots \qquad \cdots\cdots$$

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!m}{(1+x)^n}, \quad f^n(0) = (-1)^{n-1}(n-1)!m,$$

$$f^n(\theta x) = \frac{(-1)^{n-1}(n-1)!m}{(1+\theta x)^n};$$

$$\therefore \quad \log_a(1+x)$$

$$= m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \left(\frac{x-\theta x}{1+\theta x}\right)^n \cdot \left(\frac{(-1)^{n-1}}{1-\theta}\right)\right).$$

When $x < -1$, $\log_a(1+x)$ has no real value.

When $x = -1$, the odd derivatives are discontinuous.

When $x > -1$, $f(x)$ and all its successive derivatives are continuous.

Cauchy's form of remainder,

$$R_n(x) = f^n(\theta x)\frac{x^n(1-\theta)^{n-1}}{(n-1)!},$$

gives
$$R_n(x) = \left(\frac{x-\theta x}{1+\theta x}\right)^n \cdot \frac{(-1)^{n-1}}{1-\theta} m,$$

in which the second factor is finite, and the first factor $\doteq 0$, when $n = \infty$ and $x > -1$ and < 1 or $x = 1$.

Hence the *logarithmic series*

$$\log_a(1+x)$$

$$= m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-2}x^{n-1}}{n-1} + \cdots\right) \quad (1)$$

is the expansion of $\log_a(1+x)$ when $x > -1$ and < 1 or $x = 1$.

Putting $-x$ for x in (1), gives

$$\log_a(1-x) = m\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right). \quad (2)$$

Subtracting (2) from (1), gives

$$\log_a\frac{1+x}{1-x} = 2m\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \cdot \left(\begin{matrix}\text{Compare (3),}\\ \text{Ex. 1, Art. 159.}\end{matrix}\right) \quad (3)$$

Let
$$x = \frac{1}{2z+1}; \quad \text{then } \frac{1+x}{1-x} = \frac{z+1}{z}. \quad (4)$$

Substituting in (3) the values in (4), gives

$$\log_a \frac{z+1}{z} = 2m\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \cdots\right); \qquad (5)$$

$$\therefore \quad \log_a(z+1) = \log_a z + 2m\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \cdots\right). \qquad (6)$$

When $z > 0$, $0 < x < 1$; hence the series in (6) is convergent for all positive values of z.

When 1 is put for z, $\log_a 2$ is found; then 2 for z, and $\log_a 3$ is found; thus $\log_a(z+1)$ can be readily computed when $\log_a z$ is known. (See Ex. 1, Art. 159.)

When $m = 1$, $a = e$, the Napierian base; thus (5) becomes

$$\log \frac{z+1}{z} = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \cdots\right). \qquad (7)$$

Dividing (5) by (7) and denoting $\frac{z+1}{z}$ by N, gives

$$\log_a N/\log N = m; \text{ that is, } \log_a N = m \log N. \qquad (8)$$

Value of m. — Putting $N = a$ in (8), gives

$$1 = m \log a; \text{ that is, } m = 1/\log a, \qquad (9)$$

or $N = e$, gives, $\log_a e = m$;

$$\therefore \quad 1/\log a = \log_a e, \quad \text{or} \quad 1 = \log a \cdot \log_a e. \qquad (10)$$

Value of M. — Denoting the modulus of the common system whose base is 10 by M; from (9) and (10),

$$M = \frac{1}{\log 10} = \log_{10} e$$

$$= \frac{1}{2.302585} = 0.434294 \ldots . \quad \text{(Compare Art. 38.)}$$

Note. — When $a = 10$ and $m = M$, or when $a = e$ and $m = 1$, all the series in this example become *common* logarithmic series, or *Napierian* logarithmic series, respectively.

EXERCISE XXX.

1. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots.\quad \left(|x| < \frac{\pi}{2}\right).$

2. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots.\quad \left(|x| < \frac{\pi}{2}\right).$

3. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots.$

4. $\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots.$

5. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$

6. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots,$ by replacing x by $-x$ in 5.

7. $\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,$ (by combining terms of 5 and 6.)

8. $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$ by combining terms of 5 and 6, or by differentiating terms of 7.

9. $\tanh x = \frac{\sinh x}{\cosh x} = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots.$

10. From 5 and (1) and (2) of Examples 1 and 2, making

$$x = \sqrt{-1}\cdot x = ix,$$

get
$$e^{ix} = \cos x + i\sin x, \tag{1}$$
and
$$e^{-ix} = \cos x - i\sin x. \tag{2}$$

Note. — Putting π for x in (1) gives the remarkable relation, $e^{i\pi} = -1$; while putting -2π for x in (2) or 2π for x in (1) gives $e^{2i\pi} = 1$, whence $e^{i\pi} = \pm 1$, hence $i\pi$ is an imaginary value of log 1, the real value being zero.

11. From (1) and (2) of 10, by subtraction and by addition get

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad (3), \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}. \quad (4)$$

12. Evaluate $\int_0^x e^{-x^2}dx = \int_0^x (1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots)\,dx.$

Get result when end value is 1. When end value is ∞ the value of the integral is $\frac{1}{2}\sqrt{\pi}$.* This integral is important in the theory of probability.

* Williamson's *Integral Calculus,* Ex. 4, Art. 116, also Gibson's *Calculus,* Ex. 3, Art. 136.

155. The Binomial Theorem.—I. The binomial theorem is seen to be a special case of Taylor's Theorem by expanding $(x + h)^m$ in a power series in h by Taylor's formula. Thus,

$$f(x+h) = (x+h)^m; \qquad \therefore \quad f(x) = x^m,$$
$$f'(x) = mx^{m-1}, \qquad f''(x) = m(m-1)x^{m-2},$$
$$f'''(x) = m(m-1)(m-2)x^{m-3},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f^{n-1}(x) = m(m-1)\ldots(m-n+2)x^{m-n+1}.$$

Substituting these values in Taylor's series (5), of Art. 150,

$$(x+h)^m = x^m + mx^{m-1}h + \frac{m(m-1)}{2!}x^{m-2}h^2$$
$$+ \frac{m(m-1)(m-2)}{3!}x^{m-3}h^3 + \cdots$$
$$\cdots + \frac{m(m-1)\ldots(m-n+2)}{(n-1)!}x^{m-n+1}h^{n-1} + \cdots \qquad (1)$$

is the resulting Binomial Theorem.

Here $f^n(x) = m(m-1)\ldots(m-n+1)x^{m-n}$.

Hence, $f(x)$ and all its successive derivatives are continuous for all values of x.

Cauchy's form of remainder, $R_n(x) = f^n(x+\theta h)\dfrac{h^n(1-\theta)^{n-1}}{(n-1)!}$,

gives

$$R_n(x) = \frac{m(m-1)\ldots(m-n+1)}{(n-1)!}\cdot\left(\frac{h-\theta h}{x+\theta h}\right)^n\cdot\frac{(x+\theta h)^m}{1-\theta}.$$

When $|x| > h$ and $n = \infty$, the product of the first and second factors $\doteq 0$, and the last factor is finite; hence, $\underset{n=\infty}{R(x)} \doteq 0$.

Hence, the binomial theorem holds true when the first term of the binomial is greater absolutely than the second.

When m is a positive integer, the series (1) stops with the $(m+1)$th term, since $f^n(x) = 0$ when $n > m$, and is therefore a finite series of $m + 1$ terms.

If $|h| > x$, the expansion may be a power series in x; thus,

$$(h + x)^m = h^m + mh^{m-1}x + \frac{m(m-1)h^{m-2}}{2!}x^2 + \cdots$$
$$\cdots + \frac{m(m-1)\ldots(m-n+2)h^{m-n+1}}{(n-1)!}x^{n-1} + \ldots \quad (2)$$

is a true expansion when $|h| > x$.

II. $(1 + x)^m$ may be expanded in a power series in x by Maclaurin's Theorem, giving

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$
$$\cdots + \frac{m(m-1)\ldots(m-n+2)}{(n-1)!}x^{n-1} + \cdots. \quad (3)$$

As in case I, when m is a positive integer, the series (3) stops with the $(m + 1)$th term and is therefore a finite series of $m + 1$ terms. If m is negative or fractional, the series is infinite. The ratio test shows that the infinite series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$; therefore $R_n(x)$ needs examination only for $|x| \leqq 1$.

Here $f^n(x) = m(m-1)\ldots(m-n+1)(1+x)^{m-n}$,

$\therefore \quad f^n(\theta x) = m(m-1)\ldots(m-n+1)(1+\theta x)^{m-n}$.

Cauchy's form of remainder, $R_n(x) = f^n(\theta x)\dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}$,

gives

$$R_n(x) = \frac{(m-1)\cdots(m-n+1)}{(n-1!)}x^{n-1}\cdot\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\cdot mx(1+\theta x)^{m-1}.$$

For values of x between 0 and ± 1, the last factor is finite for every n; the second factor is always positive and cannot exceed unity; the first factor approaches zero as a limit as n increases without limit, since it is the expression for the nth term of the convergent series

$$1 + (m-1)x + \frac{(m-1)(m-2)}{2!}x^2 + \cdots.$$

Hence, $\underset{n=\infty}{R(x)} \doteq 0$, and the infinite series converges to $(1+x)^m$ for every value of m, when $|x| < 1$.

For $x = \pm 1$, the following results may be found proved in Chrystal's Algebra. These cases are not so important.

When $x = +1$, the series converges absolutely if $m > 0$, but conditionally if $0 > m > -1$, oscillates if $m = -1$, and diverges if $m < -1$.

When $x = -1$, the series converges absolutely if $m > 0$, and diverges if $m < 0$.

If $a > b$, $(a+b)^m$ can be written $a^m(1 + b/a)^m$ and expanded by Maclaurin's formula, since $b/a < 1$ may take the place of x in $(1+x)^m$. Hence, in this case,

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots, \quad (4)$$

which agrees with (1) and is the Binomial Theorem, proved true for $a > b$ whether m is positive or negative, whole or fractional.

If $a < b$, interchange them in (4) and the result will agree with (2) and be a true expansion of $(b+a)^m$.

156. Approximation Formulas. — Often a function may be replaced by another having approximately the same numerical value but a form better adapted for computation. In such cases the given function may be expanded in a series and a certain number of terms, beginning with the first, taken as an approximate value of the function; the number of the terms taken being according to the precision desired for the result.

The *binomial theorem* furnishes one of the most useful of the approximation formulas. Thus, if m denotes a small fraction, expanding $(1 \pm m)^n$ gives

$$(1 \pm m)^n = 1 \pm nm + \frac{n(n-1)}{2!}m^2 \pm \cdots,$$

where, since m is small, neglecting powers higher than the first, the approximate relation,

$$(1 \pm m)^n = 1 \pm nm \tag{1}$$

results. For the special case $n = \frac{1}{2}$,

$$\sqrt{1 \pm m} = 1 \pm \tfrac{1}{2}m. \tag{2}$$

For b small in comparison with a, the general form is

$$\sqrt{a^2 \pm b} = a\left(1 \pm \frac{b}{2a^2}\right). \tag{3}$$

For *examples:* $\sqrt{1+x} = 1 + \frac{1}{2}x - \cdots;$

$$\frac{1}{1+x} = 1 - x + \cdots; \quad \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \cdots.$$

For *extraction of roots in general,*

$$(a^n \pm b)^{\frac{1}{n}} = a\left(1 \pm \frac{b}{a^n}\right)^{\frac{1}{n}} = a\,(1 \pm x)^{\frac{1}{n}}, \tag{4}$$

where $x = \dfrac{b}{a^n}$. Expanding $(1 \pm x)^{\frac{1}{n}}$ gives

$$(1 \pm x)^{\frac{1}{n}} = 1 \pm \frac{1}{n}x - \frac{(n-1)}{n^2}\frac{x^2}{2!} \pm \frac{(n-1)(2n-1)}{n^3}\frac{x^3}{3!} - \cdots. \tag{5}$$

Example. — $\sqrt[5]{1000} = \sqrt[5]{1024 - 24} = 4\,(1 - \frac{3}{128})^{\frac{1}{5}}$.

Substituting $\frac{3}{128}$ for x in the series

$$(1 - x)^{\frac{1}{5}} = 1 - \frac{x}{5} - \frac{4}{5}\frac{x^2}{10} - \frac{4}{5}\frac{9}{10}\frac{x^3}{15} - \cdots,$$

gives to six figures 0.995268; hence,

$$\sqrt[5]{1000} = 4 \times 0.995268 = 3.981072.$$

Since $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$, when x is small

$$e^x = 1 + x \tag{6}$$

is the approximate relation.

From the series for $\sin x$, $\cos x$, and $\log(1+x)$,

$$\sin x = x\left(1 - \tfrac{1}{6}x^2\right), \tag{7}$$

$$\cos x = 1 - \tfrac{1}{2}x^2, \tag{8}$$

$$\log(1+x) = x - \tfrac{1}{2}x^2, \tag{9}$$

are the approximate relations when x is small.

When x is small compared with a,

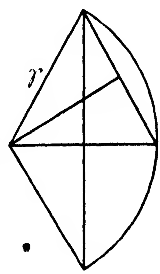
$$\sin(a \pm x) = \sin a \pm x \cos a, \tag{10}$$

$$\log(a+x) = \log a + \frac{x}{a} - \frac{x}{2a^2}, \tag{11}$$

$$\frac{1}{a \pm x} = \frac{1}{a} \mp \frac{x}{a^2} + \frac{x^2}{a^3}, \tag{12}$$

are approximate relations when succeeding terms of the respective series are neglected.

In all these cases the error made in taking the approximation for the value of the function may be closely estimated from the value of $R_n(x)$, the *error term*, for the particular series employed.



157. *Example.* — In considering the length of a circular arc and its corresponding chord in railway surveying, use may be made of the approximate relation (7). Thus, letting s denote length of the arc, r the radius, c the chord, α the angle in radians; $s = r\alpha$ and $c = 2r\sin\frac{\alpha}{2}$. When α is small,

$$c = 2r\sin\frac{\alpha}{2} = 2r\frac{\alpha}{2}\left[1 - \frac{1}{6}\left(\frac{\alpha}{2}\right)^2\right]$$

$$= r\alpha - \tfrac{1}{24}r\alpha^3; \quad \therefore \quad s - c = \tfrac{1}{24}r\alpha^3,$$

or for α in degrees, $s - c = \dfrac{r\alpha^3}{4514180}$. Since

$$R_n(x) \leqq 2r \cdot \frac{1}{120}\left(\frac{\alpha}{2}\right)^5,$$

the error of the approximation cannot exceed $\dfrac{r\alpha^5}{1920}$.

EXERCISE XXXI.

1. Expand $(x+y)^m$.
2. Expand $(x+y)^6$.
3. Expand e^{x+h}.
4. Expand $\log \sin (x+h)$.
5. Expand $\sin^{-1}(x+h)$.
6. Expand $e^{\sin x}$.
7. Given $f(x) = x^3 - 4x + 7$, find $f(x+3)$ and $f(x-2)$ by Taylor's series. Then find $f(x+3)$ and $f(x-2)$ by usual algebraic method and thus verify results.
8. Using the approximation formula (12) compute the reciprocal of 101; and of 99. Compare results with those obtained by division.
9. Find the length of the chord of an arc of radius 5729.65 feet subtending an angle of 1°: (*a*) by trigonometric methods; (*b*) by the approximation formula (7). Find results when the radius is 5729.58 feet. Compare results and find error of approximation.
10. Find the length along the slope of a road that rises 5 ft. in a horizontal distance of 100 ft. by the approximation formula (3). Determine to how many places of decimals is the result correct.

158. Application of Taylor's Theorem to Maxima and Minima. — This Article is supplementary to Art. 83, being an additional proof of the rule given in that Article for the determination of whether a critical value $x = a$, a root of $f'(x) = 0$, makes $f(x)$ a maximum, a minimum, or neither.

Let $f(x)$ be a function of x such that $f(a+h)$ and $f(a-h)$ can be expanded in Taylor's series, and let $f(a)$ be the value to be tested.

Developing $f(a-h)$ and $f(a+h)$ by formula (2), Art. 147:

$$f(a-h) = f(a) - hf'(a) + \frac{h^2}{2!}f''(a) - \frac{h^3}{3!}f'''(a) + \cdots + \frac{(-h)^n}{n!}f^n(a-\theta_1 h), \quad (1)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{(h)^n}{n!}f(a+\theta_2 h), \quad (2)$$

in which θ_1 and θ_2 are between 0 and 1 in value.

When the first $n - 1$ derivatives of $f(x)$ are zero for $x = a$ and the nth derivative is not zero for $x = a$, then,

$$f(a - h) - f(a) = \frac{(-h)^n}{n!} f^n(a - \theta_1 h), \qquad (3)$$

$$f(a + h) - f(a) = \frac{h^n}{n!} f^n(a + \theta_2 h). \qquad (4)$$

Since $f(x)$ and its successive derivatives are assumed to be continuous at and near $x = a$, the signs of $f^n(a - \theta_1 h)$ and $f^n(a + \theta_2 h)$, for very small values of h, are the same as the sign of $f^n(a)$.

It is manifest that if n is an *even* integer, $f(a)$ will be a maximum or a minimum according as $f^n(a)$ is negative or positive; and if n is *odd*, $f(a)$ will be neither a maximum nor a minimum whether the sign of $f^n(a)$ is negative or positive.

These conclusions are manifest because when n is *even* and $f^n(a)$ is negative, the left members of (3) and (4) are both negative, and hence $f(a) > f(a - h)$, $f(a) > f(a + h)$; that is, $f(a)$ is a maximum.

When n is *even* and $f^n(a)$ is positive, the left members are both positive, and hence $f(a) < f(a - h)$, $f(a) < f(a + h)$; that is, $f(a)$ is a minimum.

When n is *odd*, whether $f^n(a)$ is negative or positive, the left members have different signs, and hence $f(a) \gtrless f(a - h)$, $f(a) \lessgtr f(a + h)$; that is, $f(a)$ is neither a maximum nor a minimum regardless of the *sign* of $f^n(a)$.

159. Integration and Differentiation of Series. — A power series has the important property that, when the variable of the function is restricted to the interval of convergence, it is possible to get the integral or the derivative of the function by integrating or differentiating term by term the series which defines the function. Hence, if $f(x)$ is defined by the power series,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots; \qquad (1)$$

then

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} a_0\,dx + \int_{x_0}^{x_1} a_1 x\,dx + \cdots + \int_{x_0}^{x_1} a_n x^n\,dx + \cdots,$$

and

$$\frac{df(x)}{dx} = \frac{d(a_0)}{dx} + \frac{d(a_1 x)}{dx} + \cdots + \frac{d(a_n x^n)}{dx} + \cdots, \quad .$$

when the restriction necessary to insure convergence is placed upon the value of x.

Example 1. — For $-1 < x < 1$,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots. \tag{2}$$

Hence,

$$\int_0^x \frac{dx}{1+x} = \int_0^x dx - \int_0^x x\,dx + \int_0^x x^2\,dx - \int_0^x x^3\,dx + \cdots,$$

that is, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$ (1)

This is *the logarithmic series* with base e, and is true for $-1 < x \leqq 1$; that is, it is true for values of x within the original interval of convergence, including the end value 1; but for the other end value -1, it decreases without limit.

On putting $x = 1$ in (1),

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = 0.69 \ldots.$$

On putting $x = -1$ in (1),

$$\log 0 = -(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots) = -\infty.$$

In the same way, for $-1 < x < 1$, the integration of

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{1}$$

gives $\log(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots,$ (2)

which may be gotten by putting $-x$ for x in (1).

This logarithmic series is true for $-1 \leqq x < 1$; that is, it is true for values of x within the interval of convergence, including the end value -1; but for the other end value 1, it decreases without limit.

For $x = -1$, (2) gives log 2 as above, and for $x = 1$ log 0 as above. Hence by neither series can the logarithm of a number greater than 2 be found. By a combination of the two series the logarithm of any number can be found.

By subtracting (2) from (1),

$$\log(1+x) - \log(1-x) = \log\frac{1+x}{1-x}$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right), \text{ for } |x| < 1. \quad (3)$$

For $x = \frac{1}{3}$,

$$\log\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \log 2 = 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \cdots\right) = 0.6931\ldots.$$

For $x = \frac{1}{2}$,

$$\log\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \log 3 = 2\left(\frac{1}{2} + \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} + \frac{1}{7\cdot 2^7} + \cdots\right) = 1.0986\ldots.$$

This series (3) converges very much more rapidly for values of x less than 1 than the series (1), which converges so slowly that 100 terms give only the first two decimals correctly for the log 2, while (3) gives four decimals correctly taking only four terms of the series. Any number may be put in the form $\frac{1+x}{1-x}$, but it is necessary to calculate directly the logarithms of the prime numbers 2, 3, 5, 7 only, as the others can be expressed in terms of these. Thus,

$$\log\frac{1+\frac{1}{4}}{1-\frac{1}{4}} = \log\frac{5}{3}, \text{ and then } \log 5 = \log\frac{5}{3} + \log 3;$$

and again,

$$\log\frac{1+\frac{1}{6}}{1-\frac{1}{6}} = \log\frac{7}{5}, \text{ and then } \log 7 = \log\frac{7}{5} + \log 5.$$

To get the common logarithms whose base is 10, multiply these natural logarithms by 0.4343 . . . , the modulus of the common system. (See Art. 38 and Ex. 6, Art. 154.)

Example 2. — For $-1 < x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots. \tag{1}$$

By differentiation,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots. \tag{2}$$

By differentiation again,

$$\frac{1}{(1-x)^3} = \frac{1}{2}(1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \cdots). \tag{3}$$

Hence, the general series,

$$\frac{1}{(1-x)^m} = (1-x)^{-m} = 1 + mx + \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)(m+2)}{3!}x^3 + \cdots. \tag{4}$$

Example 3. — For $-1 < x < 1$, by (4) of Ex. 2, or by division,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots. \tag{1}$$

Hence, $\displaystyle\int_0^x \frac{dx}{1+x^2} = \int_0^x dx - \int_0^x x^2\,dx + \int_0^x x^4\,dx - \cdots,$

that is,

$$\operatorname{arc\,tan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots. \tag{2}$$

This is *Gregory's series*, named after its discoverer, James Gregory.

Although series (1) oscillates when $x = 1$, series (2) is convergent and defines $\operatorname{arc\,tan} x$ even when $x = 1$. On putting $x = 1$,

$$\operatorname{arc\,tan} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots;$$

$$\therefore \quad \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

While the value of π may be found approximately from this series, the series converges so slowly that it is better to use other more rapidly convergent series, such as,

$$\frac{\pi}{4} = 4 \operatorname{arc} \tan \frac{1}{5} - \operatorname{arc} \tan \frac{1}{239} \text{ (Machin's Series)},$$

and

$$\frac{\pi}{4} = \operatorname{arc} \tan \frac{1}{2} + \operatorname{arc} \tan \frac{1}{3} \cdot \quad \text{(Euler's Series.)}$$

Example 4. — For $-1 < x < 1$, by (4) of Ex. 2,

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots;$$

hence,

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} = \operatorname{arc} \sin x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \cdots.$$

This series, due to Newton and used by him to compute the value of π approximately, converges rapidly for $x < 1$.

When $x = \frac{1}{2}$, this series gives

$$\operatorname{arc} \sin \frac{1}{2} = \frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 2^7} + \cdots.$$

To ten places, $\qquad \pi = 3.1415926536 \ldots.$

By means of series the value of π has been carried to 700 decimal places.

Example 5. — Given $\displaystyle\int_0^a (a^2 - e^2x^2)^{\frac{1}{2}} \frac{dx}{\sqrt{a^2 - x^2}} \cdot$ This cannot be integrated directly, but on expanding $(a^2 - e^2x^2)^{\frac{1}{2}}$ by the binomial theorem the terms of the resulting convergent series can be integrated separately. Thus,

$$(a^2 - e^2x^2)^{\frac{1}{2}} = a - \frac{e^2x^2}{2a} - \frac{e^4x^4}{8a^3} - \cdots, \quad \text{where } e < 1, \quad (1)$$

$$\therefore \quad \int_0^a (a^2 - e^2x^2)^{\frac{1}{2}} \frac{dx}{\sqrt{a^2 - x^2}} \quad \text{(See Ex. 1, Exercise XXV.)}$$

$$=a\int_0^a \frac{dx}{\sqrt{a^2-x^2}}-\frac{e^2}{2a}\int_0^a \frac{x^2\,dx}{\sqrt{a^2-x^2}}-\frac{e^4}{8a^3}\int_0^a \frac{x^4\,dx}{\sqrt{a^2-x^2}}-\cdots$$

$$=\frac{\pi a}{2}\left(1-\frac{e^2}{2^2}-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5}-\cdots\right) \tag{2}$$

When $x = a\sin\theta$,

$$\int_0^a (a^2-e^2x^2)^{\frac{1}{2}}\frac{dx}{\sqrt{a^2-x^2}} = a\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{\frac{1}{2}}\,d\theta;$$

$$\therefore \quad a\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{\frac{1}{2}}\,d\theta$$

$$=a\int_0^{\frac{\pi}{2}}(1-\tfrac{1}{2}e^2\sin^2\theta-\tfrac{1}{8}e^4\sin^4\theta-\cdots)\,d\theta \quad \left(\begin{array}{c}\text{See Ex. 6, Exer-}\\ \text{cise XXII.}\end{array}\right)$$

$$=\frac{\pi a}{2}\left(1-\left(\frac{1}{2}\right)^2e^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5}-\cdots\right). \tag{2}$$

Example 6. — Given

$$\int_0^h \frac{2a\,dx}{\sqrt{2g(h-x)(2ax-x^2)}}. \quad \text{(See Art. 236, Applied Calculus.)}$$

This does not admit of direct integration, but on expanding it into a power series in $x/2a$ the integral can be evaluated approximately. Thus,

$$\int_0^h \frac{2a\,dx}{\sqrt{2g(h-x)(2ax-x^2)}} = \sqrt{\frac{a}{g}}\int_0^h \frac{dx}{\sqrt{hx-x^2}}\left(1-\frac{x}{2a}\right)^{-\frac{1}{2}},$$

by (4) of Ex. 2,

$$=\sqrt{\frac{a}{g}}\int_0^h\left[1+\frac{1}{2}\left(\frac{x}{2a}\right)+\frac{1\cdot 3}{2\cdot 4}\left(\frac{x}{2a}\right)^2+\cdots\right]\frac{dx}{\sqrt{hx-x^2}},$$

by integrating,

$$=\pi\sqrt{\frac{a}{g}}\left[1+\left(\frac{1}{2}\right)^2\frac{h}{2a}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{h}{2a}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{h}{2a}\right)^3+\cdots\right]. \tag{1}$$

When h is small in comparison with a, all terms containing $\frac{h}{2a}$ may be neglected, and the approximate value is $\pi\sqrt{\frac{a}{g}}$.

If the given integral is put in the form

$$2\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)^{-\frac{1}{2}}\,d\phi; \qquad \text{[Art. 236, Applied Calculus.]}$$

then,

$$2\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)^{-\frac{1}{2}}\,d\phi \qquad \text{[by (4) of Ex. 2.]}$$

$$=2\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}\left(1+\frac{1}{2}k^2\sin^2\phi+\frac{1}{2}\cdot\frac{3}{4}k^4\sin^4\phi+\cdots\right)d\phi,$$

By integrating,

$$=\pi\sqrt{\frac{a}{g}}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2k^6+\cdots\right]. \quad (2)$$

When k is small, the approximate value of the integral is again $\pi\sqrt{\frac{a}{g}}$.

Note. — The integral forms in Examples 5 and 6 are called elliptic integrals.

The forms $\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ and $\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\phi}\,d\phi$ are known respectively as "elliptic integrals of the first and the second kind."

In the first kind $k=\sin\frac{\alpha}{2}$; and, in Ex. 6, α is the angle each side of the vertical through which a pendulum of length a vibrates, the approximate time of a vibration being $\pi\sqrt{\frac{a}{g}}$, as found. (See Art. 236, Applied Calculus.) Tables give values for varying values of α.

In Ex. 5, e is the eccentricity of an ellipse and θ is the complement of the eccentric angle. By taking a few terms of the final series, when e is small, an approximate value of an elliptic arc of a quadrant's length is obtained. When $e = 0$ the result is the length of a quadrant of the circumference of a circle.

160. Indeterminate Forms. — It was noted at the end of Art. 20 that the derivative of $f(x)$,

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x),$$

may be finite, zero, or non-existent, but not 0/0.

The symbol 0/0 is called an indeterminate form, and when $f(x)$ takes that form for some value of x, say a, then $f(x)$ is really undefined for $x = a$, although it may be defined for any other value of x. It is possible, however, that $f(x)$ may have a definite *limit* A when x converges to a; it is customary then to call $f(a) = 0/0$ an *indeterminate form*, and to *define* A as the *value* of $f(x)$ when $x = a$, calling it the *true value* of $f(x)$ for $x = a$.

The advantage of having this "true value" assigned by definition is that $f(x)$, being in general continuous, thereby becomes continuous up to and including the value a.

Take, for example, the function $y = \dfrac{x^2 - 4}{x - 2}$. For every value of x other than $x = 2$, the function has a definite value, but for $x = 2$ it becomes $\dfrac{4 - 4}{2 - 2} = \dfrac{0}{0}$. Since the function has no definite value when $x = 2$, the limit which the function approaches as x converges to the value 2 is assigned as the value of the function when $x = 2$. If

$$x = 2 + h, \quad \lim_{h \doteq 0} \frac{(2 + h)^2 - 4}{2 + h - 2} = \lim_{h \doteq 0} (4 + h) = 4;$$

$$\therefore \quad \lim_{x \doteq 2} \frac{x^2 - 4}{x - 2} = 4.$$

Thus the true or limiting value of this function which takes the indeterminate form 0/0 is 4.

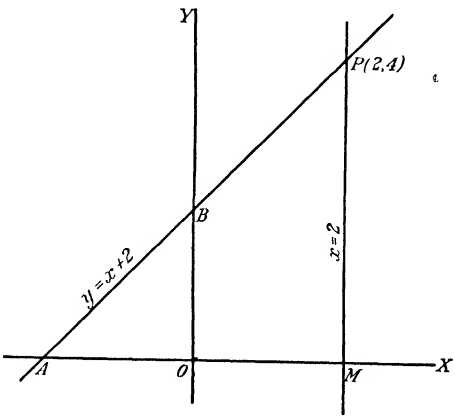
For values of x other than 2,

$$y = \frac{x^2 - 4}{x - 2} = x + 2;$$

$$\therefore \quad \left.\frac{x^2 - 4}{x - 2}\right]_{x=2} = \lim_{x \doteq 2} (x + 2) = 4.$$

On the graph of $y = x + 2$, the ordinates of points for values of x other than 2 represent the values of the function, but for $x = 2$, the function having no definite value may be represented by any ordinate lying along the line $x = 2$. Of the values that may be assigned to the function for $x = 2$, there is one value represented by $MP = 4$, which is the limit of the values represented by the ordinates of points on $y = x + 2$ as x approaches 2; and it is desirable to select this value of y as the value of the function when $x = 2$.

By this selection the function is defined for $x = 2$ and thus becomes continuous through that value of the variable x.



In general, $\lim_{x \doteq a} f(x)$ defines the value of the function when

$f(x)$ is indeterminate for $x = a$. The expression $f(x)]_a$ denotes the value of $f(x)$ when $x = a$.

The *value of a function* of x for $x = a$ usually means the result obtained by substituting a for x in the function.

When, however, the substitution results in any one of the indeterminate forms,

$$0/0, \quad \infty/\infty, \quad 0.\infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty,$$

the definition must be enlarged; thus, *the value of a function* for any particular value of its variable is the *limit* which the function approaches when the variable approaches this particular value as its limit.

This definition need be used only when the ordinary method of getting the value of the function gives rise to an indeterminate form.

161. Evaluation of Indeterminate Forms. — In many cases the limits desired are easily found by simple algebraic transformations or by the use of series. When the function that assumes the indeterminate form is the quotient of two polynomials, or can be put in that form, the following directions may be of service.

1. If the function is of the form $\dfrac{f(x)}{\phi(x)}$ and becomes $0/0$ for $x = 0$, divide both numerator and denominator by the lowest power of x that occurs in either. If the fraction becomes ∞/∞ for $x = \infty$, divide both terms of the fraction by the highest power of x in either.

2. If the function has the form $\dfrac{f(x)}{\phi(x)}$ and becomes $0/0$ for $x = a$, divide both terms by the highest power of $(x - a)$ common to both.

EXAMPLES

1. $\left.\dfrac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x}\right]_0 = -\dfrac{5}{6}.$

When $x = 0$, this fraction takes the indeterminate form $0/0$. Hence to evaluate it for $x = 0$, its limit when $x \doteq 0$ must be found. For values of x other than 0,

$$\frac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x} = \frac{x^2 + 3x - 5}{3x^3 - 2x^2 + 6};$$

$$\therefore \quad \lim_{x \doteq 0} \frac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x} = \lim_{x \doteq 0} \frac{x^2 + 3x - 5}{3x^3 - 2x^2 + 6} = -\frac{5}{6}.$$

2. $\left.\dfrac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x}\right]_\infty = 0.$

$$\frac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x} = \frac{\dfrac{1}{x} + \dfrac{3}{x^2} - \dfrac{5}{x^3}}{3 - \dfrac{2}{x} + \dfrac{6}{x^3}};$$

$$\therefore \quad \lim_{x = \infty} \frac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x} = \lim_{x = \infty} \frac{\dfrac{1}{x} + \dfrac{3}{x^2} - \dfrac{5}{x^3}}{3 - \dfrac{2}{x} + \dfrac{6}{x^3}} = \frac{0}{3} = 0.$$

3. $\left.\dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}\right]_0 = 1.$

By rationalizing numerator,

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{x}\right)\left(\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$$

$$= \frac{2}{\sqrt{1+x} + \sqrt{1-x}};$$

$$\therefore \quad \lim_{x \doteq 0} \left[\frac{\sqrt{1+x} - \sqrt{1-x}}{x}\right] = \lim_{x \doteq 0} \left[\frac{2}{\sqrt{1+x} + \sqrt{1-x}}\right] = 1.$$

4. $\left.\sqrt{1+x}-\sqrt{x}\right]_{\infty}=0.$

By changing form,

$$\sqrt{1+x}-\sqrt{x}=(\sqrt{1+x}-\sqrt{x})\frac{(\sqrt{1+x}+\sqrt{x})}{(\sqrt{1+x}+\sqrt{x})}$$

$$=\frac{1+x-x}{\sqrt{1+x}+\sqrt{x}};$$

$$\therefore \quad \lim_{x=\infty}(\sqrt{1+x}-\sqrt{x})=\lim_{x=\infty}\left[\frac{1}{\sqrt{1+x}+\sqrt{x}}\right]=0.$$

5. $\left.\dfrac{x-\sin x}{x^3}\right]_0=\dfrac{1}{6}.$

By expanding $\sin x$ in series,

$$\frac{x-\sin x}{x^3}=\frac{1}{x^3}\left[x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\right]$$

$$=\frac{1}{6}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots, \text{ if } x\neq 0;$$

$$\therefore \quad \lim_{x\doteq 0}\frac{x-\sin x}{x^3}=\lim_{x\doteq 0}\left[\frac{1}{6}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots\right]=\frac{1}{6}.$$

6. $\left.\dfrac{\sin x-x\cos x}{x^3}\right]_0=\dfrac{1}{3}.$

By expanding $\sin x$ and $\cos x$ in series,

$$\frac{\sin x-x\cos x}{x^3}=\frac{1}{x^3}\left[\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\right.$$
$$\left.-x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\right].$$

$$\therefore \quad \lim_{x\doteq 0}\left[\frac{\sin x-x\cos x}{x^3}\right]=\lim_{x\doteq 0}\left[\frac{1}{x^3}\left(\frac{x^3}{3}-\frac{x^5}{30}+\cdots\right)\right]$$
$$=\lim_{x\doteq 0}\left[\frac{1}{3}-\frac{x^2}{30}+\cdots\right]=\frac{1}{3}.$$

162. Method of the Calculus. — For the form $0/0$, to which all other indeterminate forms may be reduced, Taylor's Theorem furnishes a general method of evaluation.

I. When $f(x)$ and $\phi(x)$ are continuous functions of x and $\frac{f(x)}{\phi(x)}$ reduces to the form $0/0$ for $x = a$, the value of $\lim\limits_{x \doteq a}\left[\frac{f(x)}{\phi(x)}\right]$ is desired.

$$\text{If} \quad \left[\frac{f(x)}{\phi(x)}\right]_a = 0/0, \quad \text{then} \quad \left[\frac{f(x)}{\phi(x)}\right]_a = \left[\frac{f'(x)}{\phi'(x)}\right]_a. \tag{1}$$

That is, *if the ratio of two functions of x takes the form $0/0$ when $x = a$, then the ratio of these functions when $x = a$ is equal to the ratio of their derivatives when $x = a$.*

If $f(x+h)$ and $\phi(x+h)$ can be expanded by Taylor's formula in the neighborhood of $x = a$, it is seen that

$$\lim_{h \doteq 0}\left[\frac{f(a+h)}{\phi(a+h)}\right] = \frac{f'(a)}{\phi'(a)}; \text{ that is, } \left[\frac{f(x)}{\phi(x)}\right]_a = \left[\frac{f'(x)}{\phi'(x)}\right]_a. \tag{1}$$

By Taylor's Theorem [Art. 147, formula (2)], putting a for x,

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a+\theta_1 h)}{\phi(a) + h\phi'(a+\theta_2 h)} = \frac{f'(a+\theta_1 h)}{\phi'(a+\theta_2 h)},$$

since $f(a) = 0$, $\phi(a) = 0$; [See also (1_a), Art. 144]

$$\therefore \quad \lim_{h \doteq 0}\left[\frac{f(a+h)}{\phi(a+h)}\right] = \lim_{h \doteq 0}\left[\frac{f'(a+\theta_1 h)}{\phi'(a+\theta_2 h)}\right] = \frac{f'(a)}{\phi'(a)}. \tag{1}$$

If $f'(a)$ and $\phi'(a)$ are both zero, then [(2_a), Art. 144]

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta_3 h)}{\phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a+\theta_4 h)} = \frac{f''(a+\theta_3 h)}{\phi''(a+\theta_4 h)};$$

$$\therefore \quad \lim_{h \doteq 0}\left[\frac{f(a+h)}{\phi(a+h)}\right] = \lim_{h \doteq 0}\left[\frac{f''(a+\theta_3 h)}{\phi''(a+\theta_4 h)}\right] = \frac{f''(a)}{\phi''(a)}.$$

In this way it is seen that if, for $x = a$, $f(x)$ and $\phi(x)$ and their successive derivatives, including their nth derivatives,

are zero, while $f^{n+1}(a)$ and $\phi^{n+1}(a)$ are not both zero, then

$$\lim_{x \doteq a}\left[\frac{f(x)}{\phi(x)}\right] = \lim_{x \doteq a}\left[\frac{f^{n+1}(x)}{\phi^{n+1}(x)}\right]; \text{ that is, } \left[\frac{f(x)}{\phi(x)}\right]_a = \left[\frac{f^{n+1}(x)}{\phi^{n+1}(x)}\right]_a. \quad (2)$$

If the function $\dfrac{f(x)}{\phi(x)}$ takes the form 0/0 when x is infinite, by putting $x = \dfrac{1}{z}$ the problem is reduced to the evaluation of the limit for $z = 0$, and hence the method applies to this case also.

II. *Form* ∞/∞.—When the function $\dfrac{f(x)}{\phi(x)}$ takes the form ∞/∞, it can be reduced to the form 0/0, by writing it in the form $\dfrac{1}{\phi(x)} \Big/ \dfrac{1}{f(x)}$. This form can be evaluated as before.

Thus, let $f(a) = \infty$ and $\phi(a) = \infty$, a being finite or infinite; to show $\left[\dfrac{f(x)}{\phi(x)}\right]_a = \left[\dfrac{f'(x)}{\phi'(x)}\right]_a$.

Now $\dfrac{f(a)}{\phi(a)} = \dfrac{1}{\phi(a)} \Big/ \dfrac{1}{f(a)}$, which is in the form 0/0. Applying formula (1),

$$\left[\frac{f(x)}{\phi(x)}\right]_a = \frac{-\dfrac{\phi'(a)}{[\phi(a)]^2}}{-\dfrac{f'(a)}{[f(a)]^2}} = \left[\frac{f(a)}{\phi(a)}\right]^2 \cdot \frac{\phi'(a)}{f'(a)} = \left[\frac{f(x)}{\phi(x)}\right]_a^2 \cdot \left[\frac{\phi'(x)}{f'(x)}\right]_a;$$

$$\therefore \quad \left[\frac{f(x)}{\phi(x)}\right]_a = \frac{1}{\left[\dfrac{\phi'(x)}{f'(x)}\right]_a} = \left[\frac{f'(x)}{\phi'(x)}\right]_a.$$

If $\dfrac{f'(a)}{\phi'(a)}$ is indeterminate, continue according to formula (2) until two derivatives are obtained whose ratio is determinate, which ratio is the limiting value sought for the function.

III. *Other Forms.* — The evaluation of the other indeterminate forms may be made to depend upon the preceding.

(*a*) *Form* $0 \cdot \infty$. — When a function $f(x) \cdot \phi(x)$ takes the form $0 \cdot \infty$ for $x = a$, it may be reduced to the form $0/0$ or ∞/∞; thus,

$$f(x) \cdot \phi(x) = \frac{f(x)}{\dfrac{1}{\phi(x)}} \quad \text{or} \quad \frac{\phi(x)}{\dfrac{1}{f(x)}}.$$

(*b*) *Form* $\infty - \infty$. — By some transformation and simplification, a function taking the form $\infty - \infty$ may be reduced to a definite value, or to one of the preceding indeterminate forms.

(*c*) *Forms* 0^0, ∞^0, 1^∞. — These forms arise from a function of the form $[f(x)]^{\phi(x)}$. This function may be reduced to the form $\frac{0}{0}$. Thus let $y = [f(x)]^{\phi(x)}$, whence

$$\log y = \phi(x) \cdot \log [f(x)]. \tag{3}$$

Since for each of the given forms, (3) takes the form $0.\infty$, the evaluation is effected as in (*a*), the value of y being found from $\log y$.

EXAMPLES.

1. $\left.\dfrac{x^2 - 4}{x - 2}\right]_2 = 4.$

$$\left.\frac{x^2 - 4}{x - 2}\right]_2 = \frac{0}{0}; \quad \therefore \quad \left.\frac{x^2 - 4}{x - 2}\right]_2 = \left.\frac{2x}{1}\right]_2 = 4. \quad \text{(As in Art. 160.)}$$

2. $\left.\dfrac{x - \sin x}{x^3}\right]_0 = \dfrac{1}{6}.$

$$\left.\frac{x - \sin x}{x^3}\right]_0 = \frac{0}{0}; \quad \therefore \quad \left.\frac{x - \sin x}{x^3}\right]_0 = \left.\frac{1 - \cos x}{3x^2}\right]_0$$

$$= \left.\frac{\sin x}{6x}\right]_0 = \left.\frac{\cos x}{6}\right]_0 = \frac{1}{6}.$$

3. $\left.\dfrac{x^n - a^n}{x - a}\right]_a = na^{n-1}.$

$$\left.\frac{x^n - a^n}{x - a}\right]_a = \frac{0}{0}; \qquad \therefore \quad \left.\frac{x^n - a^n}{x - a}\right]_a = \left.\frac{nx^{n-1}}{1}\right]_a = na^{n-1}.$$

4. $\left.\dfrac{a^x - b^x}{x}\right]_0 = \log\dfrac{a}{b}.$

$$\left.\frac{a^x - b^x}{x}\right]_0 = \frac{0}{0}; \qquad \therefore \quad \left.\frac{a^x - b^x}{x}\right]_0 = \log a \cdot a^x - \log b \cdot b^x\Big]_0$$

$$= \log a - \log b = \log\frac{a}{b}.$$

5. $\left.\dfrac{e^x - e^{-x} - 2x}{x - \sin x}\right]_0 = 2.$

$$\left.\frac{e^x - e^{-x} - 2x}{x - \sin x}\right]_0 = \frac{0}{0}; \qquad \therefore \quad \left.\frac{e^x - e^{-x} - 2x}{x - \sin x}\right]_0 = \left.\frac{e^x + e^{-x} - 2}{1 - \cos x}\right]_0$$

$$= \left.\frac{e^x - e^{-x}}{\sin x}\right]_0 = \left.\frac{e^x + e^{-x}}{\cos x}\right]_0 = 2.$$

6. $\left.x\log\left(1 + \dfrac{a}{x}\right)\right]_\infty \doteq a.$

$$\lim_{x=\infty}\left[x\log\left(1+\frac{a}{x}\right)\right] = \left.\frac{\log(1+az)}{z}\right]_0 = \left.\frac{a}{1+az}\right]_0 = a, \text{ where } z = \frac{1}{x}.$$

7. $\left.\left(1 + \dfrac{a}{x}\right)^x\right]_\infty \doteq e^a; \quad \therefore \quad \left.\left(1 + \dfrac{1}{x}\right)^x\right]_\infty \doteq e.$

$$\left.\log\left(1 + \frac{a}{x}\right)^x\right]_\infty = \left.x\log\left(1 + \frac{a}{x}\right)\right]_\infty \doteq a \text{ (by Ex. 6)};$$

$$\therefore \quad \left.\left(1 + \frac{a}{x}\right)^x\right]_\infty \doteq e^a; \quad \therefore \quad \left.\left(1 + \frac{1}{x}\right)^x\right]_\infty \doteq e. \quad \begin{pmatrix}\text{Compare}\\ \text{Art. 34.}\end{pmatrix}$$

8. $\left.(1 + x)^{\frac{1}{x}}\right]_0 \doteq e.$

$$\left.(1+x)^{\frac{1}{x}}\right]_0 = 1^\infty; \quad \therefore \quad \left.\log(1+x)^{\frac{1}{x}}\right]_0 = \left.\frac{1}{x}\cdot\log(1+x)\right]_0 = \left.\frac{1}{1+x}\right]_0 = 1;$$

$$\therefore \quad \left.(1 + x)^{\frac{1}{x}}\right]_0 \doteq e. \quad \text{(Compare } Cor.\text{, Art. 34.)}$$

9. $\left.(1 - x)^{\frac{1}{x}}\right]_0 \doteq \dfrac{1}{e}$ or e^{-1}.

$$\log(1-x)^{\frac{1}{x}}\Big]_0 = \frac{1}{x}\cdot\log(1-x)\Big]_0 = \frac{-1}{1-x}\Big]_0 = -1.$$

$$\therefore \quad (1-x)^{\frac{1}{x}}\Big]_0 \doteq e^{-1} \text{ or } \frac{1}{e}.$$

EXERCISE XXXII.

Evaluate the following indeterminate forms:

1. $\dfrac{1-\cos x}{x^2}\Big]_0 = \dfrac{1}{2}.$

2. $\sec x - \tan x\Big]_{\frac{\pi}{2}} = 0.$

3. $\dfrac{\tan x - \sin x}{\sin^3 x}\Big]_0 = \dfrac{1}{2}.$

4. $(\sin x)^{\tan x}\Big]_{\frac{\pi}{2}} = 1.$

5. $x^{\sin x}\Big]_0 = 1.$

6. $\sin x \log x]_0 = 0.$

7. $x^x]_0 = 1.$

8. $(1+x^2)^{\frac{1}{x}}\Big]_0 = 1.$

9. $\dfrac{x-1}{x^n-1}\Big]_1 = \dfrac{1}{n}.$

10. $\left(\dfrac{\sin nx}{x}\right)^m\Big]_0 = n^m.$

11. $\dfrac{\log x}{x-1}\Big]_1 = 1.$

12. $\dfrac{e^x - e^{-x}}{\sin x}\Big]_0 = 2.$

13. $\dfrac{x - \sin^{-1} x}{\sin^3 x}\Big]_0 = -\dfrac{1}{6}.$

14. $\dfrac{\tan x - x}{x - \sin x}\Big]_0 = 2.$

15. $(\log x)^x]_0 = 1.$

16. $x^{\frac{1}{1-x}}\Big]_1 = e^{-1}.$

163. Evaluation of Derivatives of Implicit Functions. —

1. Find the slope of $x^4 - a^2xy + b^2y^2 = 0$ at $(0,0)$. Here

$$\frac{dy}{dx}\Big]_{0,0} = \frac{4x^3 - a^2y}{a^2x - 2b^2y}\Big]_{0,0} = \frac{0}{0}. \quad \text{(Art. 105).}$$

Hence
$$\frac{dy}{dx}\Big]_{0,0} = \frac{12x^2 - a^2\dfrac{dy}{dx}}{a^2 - 2b^2\dfrac{dy}{dx}}\Bigg]_{0,0} = \frac{-a^2\dfrac{dy}{dx}}{a^2 - 2b^2\dfrac{dy}{dx}}\Bigg]_{0,0};$$

$$\therefore \quad \frac{dy}{dx}\left(a^2 - 2b^2\frac{dy}{dx}\right) + a^2\frac{dy}{dx}\Big]_{0,0} = 0; \text{ whence } \frac{dy}{dx}\Big]_{0,0} = 0 \text{ or } \frac{a^2}{b^2}.$$

2. Find the slope of $x^3 - 3axy + y^3 = 0$ at $(0,0)$.

Ans. $\dfrac{dy}{dx}\Big]_{0,0} = 0$ or ∞.

CHAPTER IV.

INTEGRATION AS THE LIMIT OF A SUM. SURFACES AND VOLUMES.

164. Limit of a Sum. — A definite integral has been defined (Art. 128) as an increment of an indefinite integral.

It will now be shown that *a definite integral equals the limit of the sum of an infinite number of infinitesimal increments or differentials.*

Many problems in pure and applied mathematics can be brought under the following form:

Given a continuous function, $y = f(x)$, *from* $x = a$ *to* $x = b$. *Divide the interval from* $x = a$ *to* $x = b$ *into* n *equal parts, of length* $\Delta x = (b - a)/n$. *Let* $x_1, x_2, x_3, \ldots x_n$ *be values of* x, *one in each interval; take the value of the function at each of these points, and multiply by* Δx; *then form the sum:*

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x. \tag{1}$$

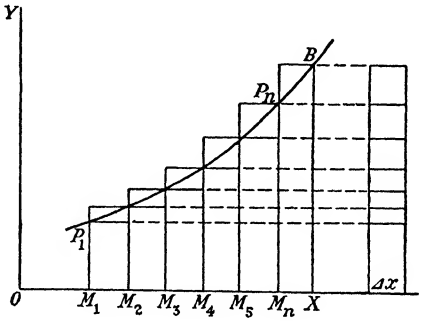
Required, the limit of this sum, as n *increases indefinitely and* Δx *approaches zero.*

This problem may be interpreted geometrically as the problem of finding the area under the curve $y = f(x)$, between the ordinates $x = a$ and $x = b$; each term of the sum representing the area of a rectangle whose base is Δx and whose altitude is the height of the curve at one of the points selected.

Let the area M_1P_1BX be denoted by A; let $OM_1 = a$, $OX = b$, and P_1B be the locus of $y = f(x)$.

Let Δx be one of the equal parts of M_1X, although the parts need not be made equal provided the largest of them approaches zero when n is made to increase indefinitely.

It is easily seen that the difference between the sum of the rectangles as formed and the area A is less than a rectangle whose base is Δx and whose altitude is a constant, $f(b) - f(a)$. Since this difference approaches zero as $\Delta x \doteq 0$, the sum of either set of rectangles approaches the area A as a



limit. It is evident that the sum of the rectangles which are partly above the curve is greater than A, while the sum of those which are wholly under the curve is less than A.

By the notation of a sum, letting T be the difference,

$$A = \sum_a^b f(x)\,\Delta x \pm T, \text{ where } T, < f(b)\,\Delta x, \doteq 0, \text{ as } \Delta x \doteq 0;$$

$$\therefore \quad \operatorname*{limit}_{\Delta x \doteq 0} \sum_a^b f(x)\,\Delta x = A = \int_a^b f(x)\,dx. \qquad (2)$$

The equation (2) is true, for it has been already shown that

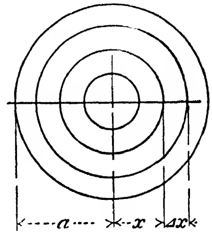
$$A = \int_a^b f(x)\,dx = \int f(x)\,dx \Big]_{x=b} - \int f(x)\,dx \Big]_{x=a} = F(b) - F(a),$$

where $\int f(x)\,dx = F(x)$. It follows that the limit of (1) is

$$\lim_{\Delta x \doteq 0} \Big[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \Big] = F(b) - F(a), \quad (3)$$

where a and b are end values of x and $\int f(x)\,dx = F(x)$

The theorem of this Article summarized in equation (2) may be said to be the *fundamental theorem* of the Integral Calculus.



As a simple example of the determination of an area by getting the limit of a sum of an indefinite number of infinitesimal elements of area, let a circle of radius a be divided into concentric rings of width Δx; then for the area A,

$$A = \underset{\Delta x \doteq 0}{\text{limit}} \sum_{x=0}^{x=a} 2\pi x \Delta x = 2\pi \int_0^a x\,dx = 2\pi x^2/2\Big]_0^a = \pi a^2.$$

Here $\quad \Delta A = 2\pi(x + \frac{1}{2}\Delta x)\Delta x$ and $dA = 2\pi x\,dx.$

165. The Summation Process. — On account of the frequency of the occurrence of the summation process, it may be said that *an integral means the limit of a sum*, the limit being in most cases most easily found as an anti-differential or anti-derivative; that is, by the inverse process to differentiation, namely, by *integration.*

The symbol $\int$ for integration, the elongated S, is derived from the initial letter of *summa*, the integral being originally conceived as a definite integral, the limit of a *sum*. According to Art. 128, the indefinite integral also may be regarded as the limit of a sum.

The fact that the summation of an indefinitely large number of indefinitely small terms is in most cases easily effected by a comparatively simple process is of the highest importance. Thus integration replaces the tedious and often difficult process of direct summation and gives an exact result, while the other often gives but an approximation at the best.

While the process of summation has been illustrated geometrically by the determination of an area, the reason

of the process by no means depends upon geometrical considerations. The method is applicable to the determination of the limit of the sum of small magnitudes of all kinds — volumes, masses, velocities, pressures, heat, work, etc. For an example of finding the limit of the sum of small volumes, consider the volume V generated by revolving the area M_1P_1BX of the figure of Art. 164, about OX as an axis. Each of the rectangles $P_1M_2, \ldots, P_nX$ will generate a cylinder whose volume will be expressed by $\pi\,(f\,(x))^2\,\Delta x$; hence,

$$V = \sum_a^b \pi\,(f\,(x))^2\,\Delta x + T,$$

where $\quad T, \; < \pi\,(f\,(b))^2\,\Delta x, \;\; \doteq 0, \;\text{ as } \Delta x \doteq 0;$

$$\therefore \quad \lim_{\Delta x \doteq 0} \sum_a^b \pi\,(f\,(x))^2\,\Delta x = V = \int_a^b \pi\,(f\,(x))^2\,dx. \qquad (1)$$

Example. — Find volume of a sphere by revolution of

$$y^2 = a^2 - x^2 : V = \int_{-a}^{a} \pi\,(a^2 - x^2)\,dx = \tfrac{4}{3}\,\pi a^3.$$

For another example of finding the limit of the sum of small volumes, find the volume of the sphere considered as made up of concentric shells of thickness $\Delta\rho$.

$$V = \lim_{\Delta\rho \doteq 0} \sum 4\,\pi\rho^2 \cdot \Delta\rho = 4\,\pi \int_0 \rho^2\,d\rho = \tfrac{4}{3}\,\pi a^3.$$

166. Approximate and Exact Summations. — When the rate of change (or the derivative) of a variable quantity is given, the total amount (or the integral of the rate) can be obtained approximately by direct summation, and exactly by finding the limit of a sum; that is, by integration.

For example, suppose the speed of a train is increasing uniformly from zero to 60 miles per hour, in 88 seconds; that is, from zero to 88 ft. per sec. in 88 seconds, the increase in speed each second (the acceleration) is 1 foot per second.

Hence the speeds at the beginnings of each of the seconds are 0, 1, 2, 3, . . . , etc.

Taking the speeds as approximately the same during each second as at the beginnings, the total distance,

$$s = 0 + 1 + 2 + 3 + \cdots + 86 + 87 = \frac{87 \cdot 88}{2} = 3828 \text{ ft.},$$

which is evidently less than the true distance.

Taking the speed at the end of a second as that during the second,

$$s = 1 + 2 + 3 + 4 + \cdots + 87 + 88 = \frac{88 \cdot 89}{2} = 3916 \text{ ft.},$$

which is evidently greater than the true distance. These values for the distance differ by 88 ft. and it is certain that the true distance is between 3828 ft. and 3916 ft. When the length of the interval during which the speed is taken as constant is reduced more and more, the result will be more and more accurate, nearer and nearer to the true distance. Manifestly, the exact distance is the *limit* approached by this summation of small distances as the interval of time Δt approaches zero:

$$s\Big]_{t=0}^{t=88} = \lim_{\Delta t \doteq 0} \sum_{t=0}^{t=88} v\,\Delta t = \int_{t=0}^{t=88} t\,dt = \frac{t^2}{2}\Big]_{t=0}^{t=88} = 3872 \text{ ft.}$$

In general,

$$s = \underset{\Delta t \doteq 0}{\text{limit}} \sum_{t=0}^{t} v\,\Delta t = \int_{t=0}^{t} at\,dt = \tfrac{1}{2}\,at^2,$$

a being constant acceleration.

In mechanics, the determinations of centers of gravity, centers of pressure, moments of inertia, varying stress, etc., involve the summation principle; and the greater number of the integrations in practice appear more naturally as *limits of sums* than as *reversed rates*, *anti-derivatives*, or *anti-differentials*.

The summation of an infinite number of terms is always

involved when one of the factors entering into the problem varies continuously. For example, in the problem of finding the mass of a body, defined as the product of density and volume; when the density ρ varies continuously,

$$m = \underset{\Delta V \doteq 0}{\text{limit}} \sum \rho \, \Delta V = \int \rho \, dV,$$

where the integral taken between "limits," that is, with end values for the independent variable, is the limit required. Hence the mass is given by a definite integral, which can be evaluated when the density ρ is a known function of the volume V, that is, of the variables x, y, z or r, θ, ϕ, in terms of which the volume may be expressed. When the density ρ is constant, it is evident that the mass is

$$m = \sum \rho \, \Delta V = \rho \int dV = \rho V.$$

Thus, when the body is composed of different liquids of varying densities in the layers or strata, the total mass is found by the addition of a finite number of terms. For if V_1, V_2, V_3, . . . V_n denote the volumes of the separate parts, and ρ_1, ρ_2, ρ_3, . . . ρ_n the corresponding densities, then

$$m = \rho_1 V_1 + \rho_2 V_2 + \rho_3 V_3 + \cdots + \rho_n V_n,$$

where the summation is made without integration.

The above will give an approximate result even when the density varies throughout the whole mass. When, however, the density varies continuously as in the atmosphere, the total volume is divided into n parts each equal to ΔV and each part is multiplied by the density at that part of the body. There are then n elements of the form $\rho \, \Delta V$, and when n is finite their summation will be an approximation to the mass of the whole; but to get the exact value, the limit of the sum, as n becomes infinite and $\Delta V \doteq 0$, must be found, and hence the exact value of the whole mass is determined by the process of integration.

Example 1. — If $y = x^2$, find $\sum_1^2 x^2\,\Delta x$ for different values of Δx, and get $\lim\limits_{\Delta x \doteq 0} \sum_1^2 x^2\,\Delta x$. Get $\lim\limits_{\Delta x \doteq 0} \sum_0^1 x^2\,\Delta x$.

When $\Delta x = 0.2$,

$$\sum_1^2 x^2\,\Delta x = (1^2 + \overline{1.2}^2 + \overline{1.4}^2 + \overline{1.6}^2 + \overline{1.8}^2)\,0.2 = 2.04.$$

When $\Delta x = 0.1$,

$$\sum_1^2 x^2\,\Delta x = (1^2 + \overline{1.1}^2 + \overline{1.2}^2 + \cdots + \overline{1.9}^2)\,0.1 = 2.18.$$

When $\Delta x = 0.05$,

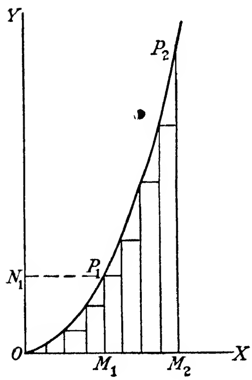
$$\sum_1^2 x^2\,\Delta x = (1^2 + \overline{1.05}^2 + \overline{1.1}^2 + \cdots + \overline{1.95}^2)\,0.05 = 2.26.$$

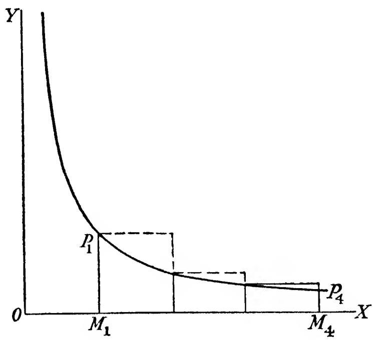
$$\lim_{\Delta x \doteq 0} \sum_1^2 x^2\,\Delta x = \int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8-1}{3}$$

$$= 2.33\tfrac{1}{3} \text{ square units in } M_1P_1P_2M_2.$$

$$\text{Lim} \sum_0^1 x^2\,\Delta x = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg]_0^1 = \frac{1}{3}$$

$$= 0.33\tfrac{1}{3} = \tfrac{1}{3} \text{ of rectangle } OM_1P_1N_1.$$





Example 2. — If $y = \dfrac{1}{x}$, find $\sum_1^4 \dfrac{\Delta x}{x}$ for different values of Δx, and get $\lim\limits_{\Delta x \doteq 0} \sum_1^4 \dfrac{\Delta x}{x}$. Get $\sum_0^1 \dfrac{\Delta x}{x}$ as $\Delta x \doteq 0$.

When $\Delta x = 1$,

$$\sum_1^4 \frac{\Delta x}{x} = \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)(1) = 1.833.$$

When $\Delta x = 0.5$, $\sum_1^4 \frac{\Delta x}{x} = 1.593.$

When $\Delta x = 0.1$, $\sum_1^4 \frac{\Delta x}{x} = 1.426.$

$$\lim_{\Delta x \doteq 0} \sum_1^4 \frac{\Delta x}{x} = \int_1^4 \frac{dx}{x} = \log x \Big]_1^4 = \log 4 - \log 1 = \log 4$$
$$= 1.386 = \text{Area } M_1P_1P_4M_4.$$

$$\lim_{\Delta x \doteq 0} \sum_a^1 \frac{\Delta x}{x} = \int_a^1 \frac{dx}{x} = \log x \Big]_a^1 = \log 1 - \log a = -\log a \Big]_{a=0} = \infty;$$

hence, when $a = 0$, the limit does not exist, as $\sum_0^1 \frac{\Delta x}{x}\Big]_{\Delta x \doteq 0} = \infty$.
(Compare Ex. 11, Art. 135.)

Note. — For examples of application see Art. 189.

EXERCISE XXXIII.

1. If $y = x$, find $\sum_3^7 x\,\Delta x$, when $\Delta x = 1$; when $\Delta x = 0.5$; when $\Delta x = 0.2$. Get $\lim_{\Delta x \doteq 0} \sum_3^7 x\,\Delta x$. *Ans.* 18; 19; 19.6.

Ans. 20.

2. If $y = \tan\theta$, find $\sum_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan\theta\,\Delta\theta$, when $\Delta\theta = \frac{\pi}{36}$; when $\Delta\theta = \frac{\pi}{60}$; when $\Delta\theta = \frac{\pi}{180}$. Get $\lim_{\Delta\theta \doteq 0} \sum_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan\theta\,\Delta\theta$. *Ans.* 0.316; 0.328; 0.340.

Ans. $\log_e \sqrt{2} = 0.346$.

Determine the following quantities (*a*) approximately by summation of a limited number of terms; (*b*) exactly by finding the limit of the sum of an infinite number of terms by integration.

3. The area under the curve $y = x^3$, from $x = 0$ to $x = 2$; from $x = -1$ to $x = 1$.

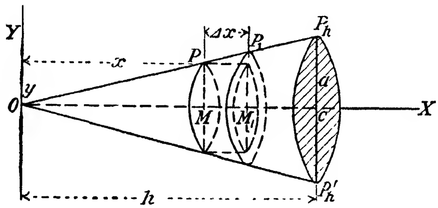
4. The distance passed over by a body falling with constant acceleration $g = 32.2$ per sec.^{2}, from $t = 1$ to $t = 4$, $v = gt$ being the relation of v and t.

5. The increase in speed of a body falling with acceleration of $g = 32.2$ per sec.², from $t = 0$ to $t = 3$.

6. The number of revolutions made in 5 minutes by a wheel which revolves with angular speed $\omega = t^2/1000$ radians per second.

7. The time required by the wheel of Ex. 6 to make the first ten revolutions.

167. Volumes. — The volumes of most solids may be found approximately by the summation of a finite number of parts and exactly by finding the limit of the sum of an infinite number of terms by integration.



Example. — To find the volume of the right circular cone whose altitude is h and the radius of whose base is a. Dividing the volume into parts, each ΔV, by passing planes Δx apart parallel to the base A_h, and denoting a section at a distance x from the vertex at the origin by A_x, then, since $A_x/A_h = x^2/h^2$, V is given approximately by

$$\sum \Delta V = \sum_0^h A_x \Delta x = \sum_0^h A_h \frac{x^2}{h^2} \Delta x \tag{1}$$

and exactly by

$$V = \lim_{\Delta x \doteq 0} \sum_0^h A_h \frac{x^2}{h^2} \Delta x = \frac{A_h}{h^2} \int_0^h x^2\, dx \tag{2}$$

$$= \frac{A_h}{h^2} \frac{x^3}{3}\Big]_0^h = \frac{1}{3} A_h h. \tag{3}$$

While ΔV is a frustum of the cone, dV may be represented by the cylinder $PMM_1 = A_x \cdot \Delta x = \pi y^2\, dx$.

It is to be noted that the equations all apply to a pyramid with any plane base A_h as well as to the cone.

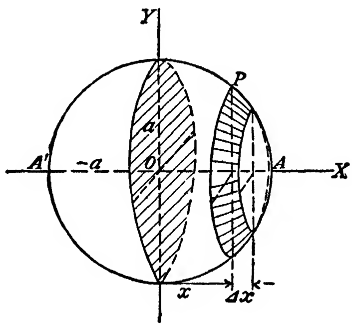
For another example: to find the volume of a sphere with radius a, divide by planes perpendicular to OX; then, since

$$\frac{A_x}{A_0} = \frac{y^2}{a^2},$$

$$\sum \Delta V = \sum A_x \Delta x = \sum A_0 \frac{a^2 - x^2}{a^2} \Delta x;$$

$$V = \lim_{\Delta x \doteq 0} \sum_{-a}^{a} A_0 \frac{a^2 - x^2}{a^2} \Delta x = \frac{A_0}{a^2} \int_{-a}^{a} (a^2 - x^2)\, dx$$

$$= \frac{A_0}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a} = \frac{A_0}{a^2} \cdot \frac{4}{3} a^3 = \frac{4}{3} \pi a^3, \text{ where } A_0 = \pi a^2.$$



Otherwise;

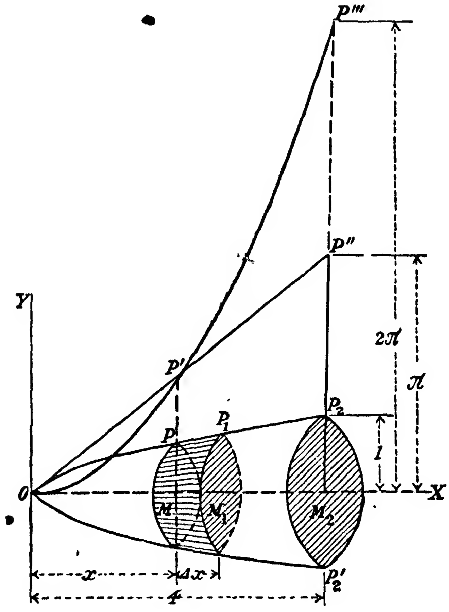
$$\sum \Delta V = \sum A_x \Delta x = \sum \pi y^2 \Delta x, \text{ where } A_x = \pi y^2;$$

$$\therefore \quad V = \lim_{\Delta x \doteq 0} \sum \pi y^2 \Delta x = \pi \int_{-a}^{a} (a^2 - x^2)\, dx = \frac{4}{3} \pi a^3.$$

168. Representation of a Volume by an Area. — In Art. 138 on the significance of an area as an integral it was stated that the integrals represented by areas might be functions of various kinds. To show an example of a volume as an integral represented by an area under a curve, let the volume of the paraboloid of revolution, between $x = 0$ and $x = 4$, be first found as the limit of the sum of the parts between

the parallel planes Δx apart, as $\Delta x \doteq 0$ and the number of the parts increases without limit. The equation of the generating parabola being $y^2 = \frac{1}{4}x$,

$$V = \lim_{\Delta x \doteq 0} \sum_{x=0}^{x=4} \pi y^2 \Delta x = \frac{\pi}{4}\int_0^4 x\,dx = \frac{\pi}{4}\left.\frac{x^2}{2}\right]_0^4 = 2\pi \text{ cubic units.}$$



To represent this volume graphically by an area, the line OP' is drawn by the equation $y = \frac{\pi}{4}x$, this being the function which was integrated to get the volume of the solid $P_2OP'_2$.

Producing the ordinate M_2P_2 to P'', the area OM_2P'' graphically represents the volume of the solid $P_2OP'_2$. For,

$$\text{Area } OM_2P'' = \int_0^4 y\,dx = \frac{\pi}{4}\int_0^4 x\,dx = \frac{\pi}{4}\left.\frac{x^2}{2}\right]_0^4 = 2\pi \text{ square units.}$$

The last result may be verified by noting that the ordinate M_2P'', for $x = 4$, being π, the area of the triangle is 2π.

In the same way it may be seen that any part of the area, as OMP', represents the corresponding part of the volume of the solid; that is, there is the same number of square units in the one as there are cubic units in the other.

If $OP'P'''$, the first integral curve of $OP'P''$, whose equation is

$$y = \int_0^x \frac{\pi}{4} x\,dx = \frac{\pi x^2}{8} \text{ (see Art. 140)}$$

be drawn, its *ordinates* will represent both the areas of the parts of OM_2P'' and the volumes of the parts of the paraboloid measured from O; that is, the measure of the ordinates in linear units will be the same as that of the areas in square units and that of the volumes in cubic units.

$$\text{Length of } M_2P''' = y = \frac{\pi x^2}{8}\Big]^{x=4} = 2\pi \text{ linear units.}$$

Note. — The volume of the cone of Art. 167, may be graphically represented by the area under the parabola $y = \frac{A_h}{h^2}x^2$, and the volume of the sphere by the area under the parabola $y = \pi(a^2 - x^2)$. If the first integral curves,

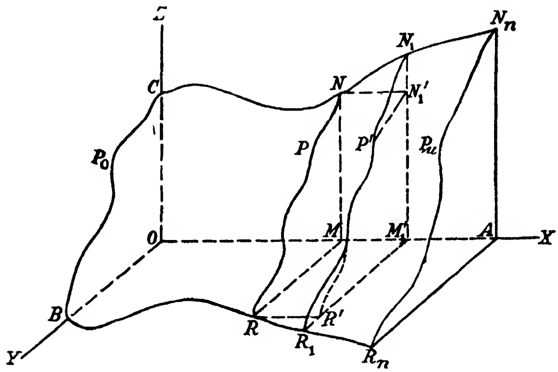
$$y = \frac{A_h}{3h^2}x^3 \quad\text{and}\quad y = \pi\left(a^2x - \frac{x^3}{3}\right),$$

be drawn, their *ordinates* will represent both the areas and the volumes in the two cases, respectively.

169. Surface and Volume of Any Frustum. — A solid bounded by two parallel planes is, in general, called a *frustum*. One or both of the truncating planes may in special cases, as in the sphere, touch the frustum in only one point and be tangent planes.

The method of dividing the solid into thin slices and taking the sum of the approximate expressions for the small parts as an approximate expression for the whole, and taking the

limit of the sum as an exact expression for the whole, may be applied to any solid even when the solid is not regular and the sections not regular plane curves.



Let the solid represented in the figure be divided into slices by planes perpendicular to an axis OX; then, taking $A_x\Delta x$ as an approximate expression for the volume of the slice $P - N_1M_1'R_1$, A_x being the area of the section $PNMR$ at a distance x from plane ZOY, the approximate expression is

$$\sum \Delta V = \sum_{x=0}^{x=h} A_x \Delta x,$$

where $h\,(= OA)$ is the distance between the truncating or bounding planes. The exact expression is

$$V = \lim_{\Delta x \doteq 0} \sum_{x=0}^{x=h} A_x \Delta x = \int_{x=0}^{x=h} A_x\,dx. \tag{1}$$

When the area of a section is a function of the distance x from one of the bounding planes and hence A_x can be expressed in terms of x, the limit may be found by integration. The frustum formula for volumes is, therefore,

$$V = \int_{x=0}^{x=h} F(x)\,dx \quad \text{or} \quad V = \int_{x=h_1}^{x=h_2} F(x)\,dx, \tag{1'}$$

where A_x is $F(x)$, some function of x; the one form giving the whole volume and the other a segment or any part thereof.

To get expressions for the area of the surface S, let P be the curve NPR, then $\Delta S = NPRR_1P'N_1$, and the approximate expression is

$$\sum \Delta S = \sum_{s=0}^{s=s_h} P\,\Delta s,$$

where $\Delta s = NN_1$ and s is the length of CN.

The exact expression for the surface is

$$S = \underset{\Delta s \doteq 0}{\text{limit}} \sum_{s=0}^{s=s_h} P\,\Delta s = \int_{s=0}^{s=s_h} P\,ds. \tag{2}$$

When the curve P is a function of s, the bounding curve in XZ plane, and can be expressed in terms of s, or when ds can be expressed in terms of P, with change of end values, the limit can be found by integration. If the surface S is conceived as generated by the curve NPR as it moves with its plane always perpendicular to OX, when its plane is in the position as shown, at a distance x from plane YZ, let NN'_1 be drawn equal to ds but *parallel* to OX; then since the surface is cylindrical, the increase of S, if the increase became *uniform*, is

$$dS = P\,ds, \quad \text{the surface } NPRR'P'N_1';$$

$$\therefore \quad S = \int_{s=0}^{s=s_h} P\,ds. \tag{2}$$

If the curve NPR is a circle, as in solids of revolution, with the center at M on the x-axis, then $P = 2\,\pi y$ and $A_x = \pi y^2$, (2) and (1) becoming

$$S = 2\pi \int_{s=0}^{s=s_h} y\,ds \tag{3}$$

and

$$V = \pi \int_{x=0}^{x=h} y^2\,dx, \tag{4}$$

where $dV = \pi y^2\,dx$ is the volume of the cylinder generated by the area of the circle πy^2, as it moves uniformly through $\Delta x = dx$.

Note. — In deriving (1) and (4), in the figure, $NN_1' = \Delta x = dx$; while in deriving (2) and (3), ds, the uniform change of s along a tangent to the curve CN at the point N, is drawn parallel to OX and represented in length by NN_1', although it is not the same as $\Delta x = dx$ but is really longer.

Example 1. — To find the lateral surface of the cone of Art. 167: by (3),

$$S = 2\pi \int_0^{s_h} y\,ds = 2\pi \int_0^{s_h=l} \frac{a}{l} s\,ds, \text{ where } s = \frac{l}{a} y = OP,$$

$$= 2\pi \frac{a}{l}\frac{s^2}{2}\Big]_0^l = \pi al, \text{ where } l = OP_h, \text{ an element.}$$

Again,

$$S = 2\pi \int_0^{s_h} y\,ds = 2\pi \int_0^{y=a} y\frac{l}{a}\,dy, \text{ since } ds = d\left(\frac{l}{a}y\right) = \frac{l}{a}\,dy,$$

$$= 2\pi \frac{l}{a}\frac{y^2}{2}\Big]_0^a = \pi al,$$

limit or end value being changed from l to a.

Example 2. — To find the surface of the paraboloid of Art. 168:

$$S = 2\pi \int_0^{s_h} y\,ds = 2\pi \int_0^{y=1} y\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy$$

$$= 2\pi \int_0^1 y\,[1 + 64\,y^2]^{\frac{1}{2}} dy, \quad \text{from } y^2 = \frac{1}{4}x,$$

$$= \frac{2\pi}{128}\left[1 + (64\,y^2)^{\frac{3}{2}}\right]_0^1 \frac{2}{3} = \frac{\pi}{96}\left((65)^{\frac{3}{2}} - 1\right)$$

$$= \frac{\pi}{96}(65\sqrt{65} - 1) \text{ square units.}$$

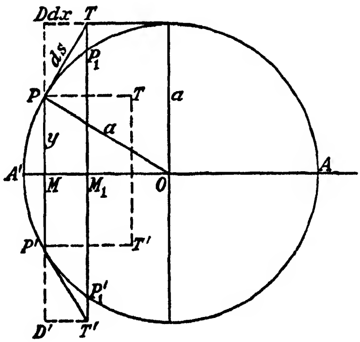
Example 3. — To find the surface of the sphere of Art. 167, or any part of it, as a zone.

For a change take origin at A' on the circumference, making $y = \sqrt{2ax - x^2}$ and $2\pi y$ the curve P bounding the section A_x; then by (2) or (3),

$$S = \int_0^{s_h} P\,ds = 2\pi \int_0^{s_h} y\,ds = 2\pi \int_{x_0}^{x} a\,dx$$

(where $y\,ds = a\,dx$, from similar triangles, OMP and PDT)

$$= 2\pi ax \Big]_{x_0}^{x} = 2\pi a(x - x_0) \quad \text{or} \quad 2\pi ax\Big]_0^{2a} = 4\pi a^2.$$



Drawing $PT = ds$ from P parallel to x-axis, $2\pi y\,ds$ is the lateral surface of the cylinder PT', which is equal in area to that of the cylinder DT', which is $2\pi a\,dx$.

The volume is again, with origin at A', by (4),

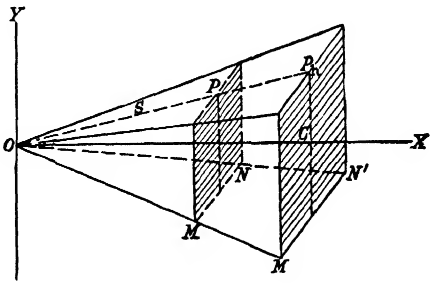
$$V = \pi \int_0^{2a} y^2\,dx = \pi \int_0^{2a} (2ax - x^2)\,dx = \pi \left[\frac{2ax^2}{2} - \frac{x^3}{3}\right]_0^{2a} = \frac{4}{3}\pi a^3.$$

Example 4 — To find the lateral surface of a quadrangular pyramid. Let P_h = perimeter of base and $l = OP_h$ = slant height. Let PMN be the position of the generating perimeter P when $s = OP$. Since P and P_h are similar,

$$\frac{P}{P_h} = \frac{OP}{OP_h} = \frac{s}{l}; \quad \text{hence,} \quad P = \frac{P_h}{l} s, \text{ in (2)};$$

$$S = \int_0^{s_h} P\,ds = \frac{P_h}{l}\int_0^l s\,ds = \frac{P_h}{l}\frac{s^2}{2}\Big]_0^l = \frac{P_h l}{2};$$

that is, the convex surface of any pyramid or cone (Ex. 1) is measured by half the product of perimeter of base and slant height.



For the volume,

$$V = \int_0^h A_x\,dx = \frac{A_h}{h^2}\int_0^h x^2\,dx, \quad \text{since} \frac{A_x}{A_h} = \frac{x^2}{h^2}, \quad A_x = \text{area } PMN,$$

$$= \frac{A_h}{h^2}\frac{x^3}{3}\Big]_0^h = \frac{1}{3}A_h h;$$

that is, the volume of any pyramid or cone ((3), Art. 167) is measured by one-third the product of its base and altitude.

Note. — The foregoing, for the purpose of illustration, have been for the most part examples of elementary solids whose surfaces and volumes are known from solid geometry. The fruitfulness of the method is seen in the determination of the surfaces and volumes of the frusta of unfamiliar and complex solids.

The following are some examples:

Example 5. A monument is to be built in horizontal rectangular sections, one side of a section to vary as the distance below the top and the other as the square of this

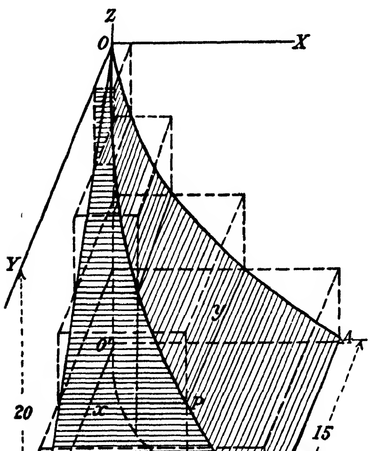
distance. The base is to be a square 30 feet on a side, and the height of the monument is to be 20 feet. Find the volume when it is made up of rectangular blocks with vertical sides; and also the volume when the sections vary continuously from top to base.

Taking origin at top and z positive downward, let $A_z = 4xy$, $2y = az$, $2x = bz^2$; $2y = 20a = 30$; $\therefore \; a = \frac{3}{2}$; $2x = 400b = 30$; $\therefore \; b = \frac{3}{40}$; $\therefore \; y = \frac{3}{4}z$, for line OB; $x = \frac{3}{80}z^2$, for curve OA. $A_z = 4xy = abz^3 = \frac{9}{80}z^3$,

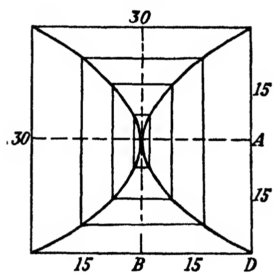
$$V = \sum_{z=0}^{z=20} \frac{9}{80} z^3 \Delta z \Big]_{\Delta z = 5} = 5 \cdot \frac{9}{80}\left[0 + 5^3 + 10^3 + 15^3 + 20^3\right]$$

$$= 7031\tfrac{1}{4} \text{ cu. ft.}$$

$$V = \lim_{\Delta z \doteq 0} \sum_{z=0}^{z=20} \frac{9}{80} z^3 \Delta z = \frac{9}{80} \int_0^{20} z^3\,dz = \frac{9}{80}\left[\frac{z^4}{4}\right]_0^{20} = 4500 \text{ cu. ft.}$$



Hence, as Δz, the thickness of the blocks, is made less and less, the volume will approach 4500 cu. ft., the volume when sections vary continuously. The plan is shown on reduced scale. The projection of OD, the curve of intersection of the



plane surface OBD and the curved surface OAD, is $O'D$ on plane parallel to XY plane. The equation of $O'D$ is

$y^2 = 15\,x$, by eliminating z from $y = \frac{3}{4}z$ and $x = \frac{3}{80}z^2$. The plan shows the corners of the blocks on this curve.

Example 6. Find the lateral surface of the monument of Ex. 5. When built of rectangular blocks, the sum of the rectangular areas gives the area of the surface. When the stone is shaped to make sections vary continuously, or when this is effected by using concrete in shaped forms, find the areas of the surfaces OBD and OAD separately.

$$S = OAD = \int P\,ds = \int_0^{20} \frac{3}{4}z\,ds = \frac{3}{4}\int_0^{20} z\left[1 + \left(\frac{dx}{dz}\right)^2\right]^{\frac{1}{2}} dz$$

$$= \frac{3}{4}\int_0^{20} z\left[1 + \left(\frac{3}{40}z\right)^2\right]^{\frac{1}{2}} dz = \frac{3}{4}\cdot\frac{1600}{18}\cdot\frac{2}{3}\left[\left(1 + \frac{9}{1600}z^2\right)^{\frac{3}{2}}\right]_0^{20}$$

$$= \frac{400}{9}\left[\frac{13}{8}\sqrt{13} - 1\right]_0^{20}.$$

$$4\,(OAD) = \frac{1600}{9}\left[\frac{46.8 - 8}{8}\right] = \frac{200}{9} \times 38.8 = 862 \text{ sq. ft.}$$

$$S = OBD = \int P\,ds = \int_0^{20} \frac{3}{80}z^2\,ds = \frac{3}{80}\int_0^{20}\left[1 + \left(\frac{dy}{dz}\right)^2\right]^{\frac{1}{2}} dz$$

$$= \frac{3}{80}\int_0^{20} z^2\left[1 + \frac{9}{16}\right]^{\frac{1}{2}} dz = \frac{3}{80}\cdot\frac{5}{4}\int_0^{20} z^2\,dz = \frac{3}{64}\left[\frac{z^3}{3}\right]_0^{20}$$

$$= \frac{3}{64}\cdot\frac{8000}{3} = 125 \text{ sq. ft.}$$

$4\,(OBD) = 4 \times 125 = 500$ sq. ft. Total surface $= 1362$ sq. ft.

Note. — Since OBD is a plane surface, its area may be found by

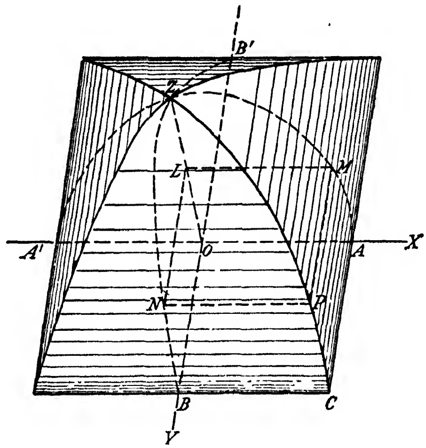
$$A = \int_0^{s_h} x\,ds = \int_{s=0}^{s=25} \frac{3}{125}s^2\,ds = \frac{3}{125}\cdot\frac{s^3}{3}\Big]_0^{25} = 125 \text{ sq. ft.,}$$

as above. Here $x = \frac{3}{125}s^2$ is the equation of curve OD in the oblique plane of sx, for since $OB = 25$, $y = \frac{3}{5}s$, and $y^2 = 15\,x$ becomes $\frac{9}{25}s^2 = 15\,x$, or $x = \frac{3}{125}s^2$.

In Ex. 5 above, $y^2 = 15\,x$ is given as equation of projection of OD on xy plane, or any plane parallel thereto.

While OD is given as a line in space by two equations, by rotating axis OY about OX through $\tan^{-1}\frac{20}{15} = \cos^{-1}\frac{3}{5}$, it is given by one equation, $s^{2\prime} = \frac{125}{3}x$, in plane of sx.

Example 7. Find the volume common to two right circular cylinders of equal radius a, whose axes intersect at right angles.



Let the two cylinders be $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$; then $A_z = LMPN = xy = a^2 - z^2$, and

$$V = 8\int_0^a A_z\,dz = 8\int_0^a (a^2 - z^2)\,dz = 8\left[a^2 z - \frac{z^3}{3}\right]_0^a = \frac{16}{3}a^3.$$

The total volume common, being 8 times $Z - OACB$, is $\frac{16}{3}a^3$.

Example 8. A dome has the shape of the figure of Ex. 7, find the area of the curved surface.

The surface ZBC is equal in area to the surface ZAC, and is one-eighth part of the surface of the dome, which surface is the upper half of the surface of the common volume of Ex. 7.

Hence the surface of the dome of eight equal parts is given by

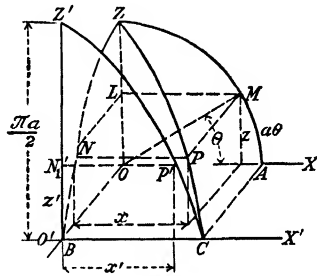
$$S = 8\,ZBC = 8\int_0^{s_h} P\,ds = 8\int_0^{s_h} NP\,ds$$

$$= 8\int_0^a x\left[1 + \left(\frac{dy}{dz}\right)^2\right]^{\frac{1}{2}} dz$$

$$= 8\int_0^a x\left[1 + \frac{z^2}{y^2}\right]^{\frac{1}{2}} dz$$

$$= 8\,a\int_0^a \frac{x}{y}\,dz = 8\,a\int_0^a dz = 8\,a^2.$$

The result shows that each of the curved surfaces of the solid $Z - OACB$ is equal in area to its base $OACB$; the surface of the dome being just twice that of its base.

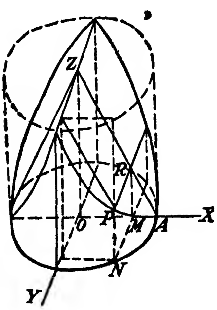


Note. — Another determination of the area of ZBC may be made by developing the curved surface upon a plane and finding the area as a plane area. Thus, developing ZBC as the plane area $Z'BC$, with B as origin;

$$\text{area } Z'BC = \text{area } ZBC = \int_0^{\frac{\pi a}{2}} x'\,dz',$$

where $x = x' = a\cos\theta$, and $z' = a\theta$

$$= \int_0^{\frac{\pi}{2}} a\cos\theta d\,(a\theta) = a^2 \sin\theta\Big]_0^{\frac{\pi}{2}} = a^2.$$



Example 9. Given a right cylinder of altitude h, and radius of base a. Through a diameter of the upper base two planes are passed, touching the lower base on opposite sides. Find the volume included between the planes.

$$V = \int_0^h A_x\,dx = 4\int_0^a (MNPR)\,dx = 4\int_0^a yz\,dx$$

$$= 4\int_0^a (a^2 - x^2)^{\frac{1}{2}}\frac{h}{a}(a - x)\,dx$$

$$= \frac{4h}{a}\int_0^a a(a^2 - x^2)^{\frac{1}{2}}\,dx - \frac{4h}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}}x\,dx$$

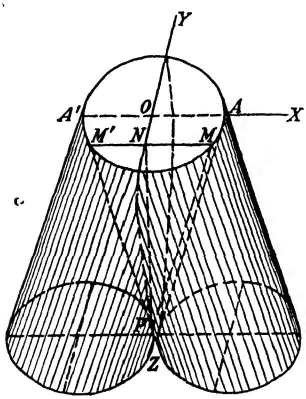
$$= 4h\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a + \frac{4h}{3a}(a^2 - x^2)^{\frac{3}{2}}\Bigg]_0^a$$

$$= \pi a^2 h - \frac{4}{3}a^2 h = (\text{vol. of cylinder}) - (\text{vol. outside the planes}).$$

Here

$$NP = MR = OZ \cdot \frac{MA}{OA}; \quad \therefore \quad NP = z = \frac{h}{a}(a - x);$$

$$MN = RP = y = (a^2 - x^2)^{\frac{1}{2}}.$$



It may be noted that, when h is equal to a, the volume outside of the planes being $\frac{4}{3}a^3$, is one-fourth of the volume common to the two cylinders of Ex. 7.

Example 10. Two cylinders of equal altitude h have a circle of radius a, for their common upper base. Their lower bases are tangent to each other. Find the volume common to the two cylinders.

$$V = \int_{h_1}^{h_2} A_y\,dy = \int_{-a}^{a} (PMM')\,dy = \int_{-a}^{a} xz\,dy$$

$$= \int_{-a}^{a} x\frac{h}{a}x\,dy = \frac{h}{a}\int_{-a}^{a} x^2\,dy = \frac{h}{a}\int_{-a}^{a}(a^2 - y^2)\,dy$$

$$= \frac{h}{a}\left[a^2y - \frac{y^3}{3}\right]_{-a}^{a} = \frac{4}{3}a^2h.$$

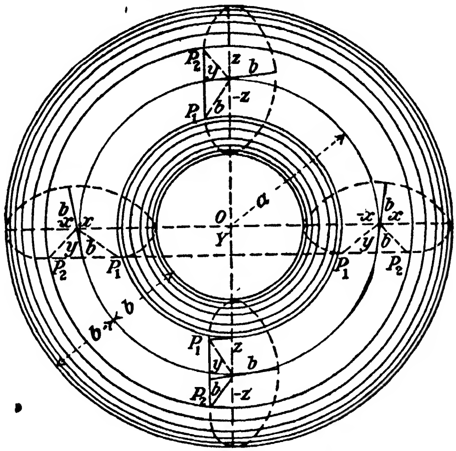
Here, PMM' being similar to ZAA',

$$NP = \frac{OZ \cdot NM}{OA}, \quad \text{or} \quad z = \frac{h}{a}x, \text{ where } x^2 = a^2 - y^2.$$

P is on curve of intersection of the cylinders.

It is seen that the volume found is equal to the volume outside the planes of Ex. 9.

Example 11. A torus is generated by a circle of radius b revolving about an axis in its plane, a being the distance of the center of the circle from the axis.



Find the volume by means of sections perpendicular to the axis.

$$V = \int_{h_1}^{h_2} A_y\,dy = \int_{x=-b}^{x=b} [\pi(a+x)^2 - \pi(a-x)^2]\,dy$$

$$= \pi \int_{y=-b}^{y=b} [(a + \sqrt{b^2 - y^2})^2 - (a - \sqrt{b^2-y^2})^2]\,dy$$

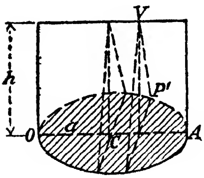
$$= \pi \int_{-b}^{b} 4\,a\sqrt{b^2 - y^2}\,dy$$

$$= 4\,\pi a \left[\frac{y}{2}\sqrt{b^2 - y^2} + \frac{1}{2}b^2 \sin^{-1}\frac{y}{b}\right]_{-b}^{b}$$

$$= 4\,\pi a \left[\frac{\pi b^2}{2}\right] = 2\,\pi^2 ab^2 = 2\,\pi a \cdot \pi b^2.$$

Note. — The last form of the result shows that the volume s the product of the area of the cross section and the length of the circumference described by the center of the revolving circle, radius a being mean of $a + b$ and $a - b$.

EXERCISE XXXIV.



1. Find the volume of the right conoid whose base is a circle of radius a, and whose altitude is h.

(*a*) With origin at 0, on the circumference; $y^2 = 2ax - x^2$.

(*b*) With origin at C, center; $y^2 = a^2 - x^2$.

Ans. $\dfrac{\pi a^2 h}{2}$.

2. An isosceles triangle moves perpendicular to the plane of the ellipse $x^2/a^2 + y^2/b^2 = 1$, its base is the double ordinate of the ellipse, and the vertical angle $2A$ is constant. Find the volume generated by the triangle. *Ans.* $\dfrac{4ab^2 \cot A}{3}$.

3. Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ by considering the volume generated by moving a variable ellipse along the axis of X. Area of ellipse $= \pi ab$. From result get volume of a sphere.

Ans. $\frac{4}{3}\pi abc$.

4. A football is 16 inches long and a plane section containing a seam is an ellipse the minor axis of which is 8 inches in length. Find the volume (*a*) if the leather is so stiff that every cross section is a square; (*b*) if the cross section is a circle. *Ans.* (*a*) $341\frac{1}{3}$ cu. in.

(*b*) $\dfrac{512\pi}{3}$ cu. in.

5. To fell a tree $2a$ feet in diameter, a cut is made halfway through from each side. The lower face of each cut is horizontal; the inclined face makes an angle of 45° with the horizontal. Find the volume of the wood cut out. Compare Ex. 9 of illustrative examples.

Ans. $\frac{4}{3} a^3$ cu. ft.

6. Find the volume of the elliptic paraboloid $2x = \dfrac{y^2}{p} + \dfrac{z^2}{q}$ cut off by the plane $x = h$. *Ans.* $\pi\sqrt{pq}\, h^2$.

7. Find the volume of Ex. 9 by moving the trapezoidal section along the Y-axis. Note that the triangular section of the volume outside the

cutting planes will at the same time generate that volume, the same as the volume of 5 above, when $h = a$.

8. A cap for a post is a solid of which every horizontal section is a square, and the corners of the square lie in the surface of a sphere 12 inches in diameter with its center in the upper face of the cap. The depth of the cap is 4 inches. Find the volume of the cap. Compare Ex. 7 of illustrative examples.

Ans. $490\frac{2}{3}$ cu. in.

9. Find the surface of the cap of 8, above. Compare Ex. 8 of illustrative examples.

Ans. Curved surface = 192 sq. in.; surface of top = 72 sq. in.

10. Show that the volume of the frustum of any pyramid or cone is equal to $\frac{h}{3}(A_0 + A_h + \sqrt{A_0 A_h})$ where A_0 and A_h are the bases, and h is its height.

170. Prismoid Formula.* — If two solids contained between the same two parallel planes have all their corresponding sections parallel to these planes equal, that is, if the area A_s' of the one is the same as the area A_s'' of the other, then their total volumes are equal, since the two volumes are given by the same integral. Let the distance between the bounding planes be, in general, $s = x$, or y, or z.

If the area A_s is a section of a solid included between two parallel planes and is a quadratic function of s,

$$A_s = as^2 + bs + c, \tag{1}$$

where s is the distance of the section A_s from one of the two parallel planes, then the volume is given by

$$V\Big]_0^{s=h} = \int_{s=0}^{s=h} (as^2 + bs + c)\,ds = \left[a\frac{s^3}{3} + b\frac{s^2}{2} + cs\right]_{s=0}^{s=h}$$
$$= \frac{ah^3}{3} + \frac{bh^2}{2} + ch, \tag{2}$$

where h is the distance of the terminal plane from the initial plane of reference; that is, the height, or length, of the solid, as the case may be.

* This derivation of the formula is substantially that given in Davis's *Calculus*.

$$\text{The area } A_0 = A_s\Big]_{s=0} = as^2 + bs + c\Big]_{s=} = c;$$

$$\text{the area } A_h = A_s\Big]_{s=h} = as^2 + bs + c\Big]_{s=h} = ah^2 + bh + c;$$

$$\text{and the area } A_m = A_s\Big]_{s=\frac{h}{2}} = as^2 + bs + c\Big]_{s=\frac{h}{2}} = \frac{ah^2}{4} + \frac{bh}{2} + c,$$

where A_m is the area of a section midway between the end sections, A_0 and A_h.

The *average* of A_0, A_h, and 4 times A_m, is

$$\frac{1}{6}(A_0 + A_h + 4A_m) = \frac{ah^2}{3} + \frac{bh}{2} + c;$$

and this average section multiplied by h is the total volume:

$$V\Big]_{s=0}^{s=h} = \frac{A_0 + A_h + 4A_m}{6} \times h = \frac{ah^3}{3} + \frac{bh^2}{2} + ch, \text{ as in (2)}. \quad . \quad . \quad . \quad (3)$$

This is the *Prismoid Formula,* so called because it holds *not only for every solid whose volume is given in elementary geometry but for any prismoid,* that is, for a solid with any end sections whatever, with sides formed by straight lines joining points of one end section with points of the other end section.

171. Application of the Prismoid Formula. — The formula holds even for many solids that are not prismoids, for example, spheres and paraboloids. It holds for all solids defined by equation (1), Art. 170, and even for all cases where A_s is any cubic function of s:

$$A_s = as^3 + bs^2 + cs + d. \qquad (1)$$

When $f(x)$ is a quadratic or a cubic function of x; then, in general,

$$\int_{x=a}^{x=b} f(x)\,dx = \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\frac{b-a}{6}, \qquad (2)$$

in accordance with the prismoid formula. The practical application of the formula is mainly for the close approximation it gives to the volume of objects in nature; for any

elevation or irregularity of the crust of the earth can be approximated to quite closely, either by the frustum of a cone, sphere, cylinder, pyramid, paraboloid, wedge, or prism; and as the formula holds for these solids as well as for any combination of them, it can be applied without determining which of the solids actually approximates most nearly to the object whose volume is desired. While it is thus used to approximate to the volumes of irregular solids, it is to be remembered that it gives the *exact* volume, when the area of a section A_s is either a quadratic or a cubic function of s, including of course a linear function as a special case of the quadratic or cubic function.

Example 1. — In the case of the cone or pyramid, Art. 167, it is seen that $A_x = A_h \frac{x^2}{h^2}$ is a quadratic function of x, and

hence $$V = \frac{1}{6}\left(0 + A_h + 4\frac{A_h}{4}\right)h = \frac{1}{3}A_h h.$$

Example 2. — In the case of the sphere,

$$A_x = A_0 \frac{a^2 - x^2}{a^2} = \pi (a^2 - x^2),$$

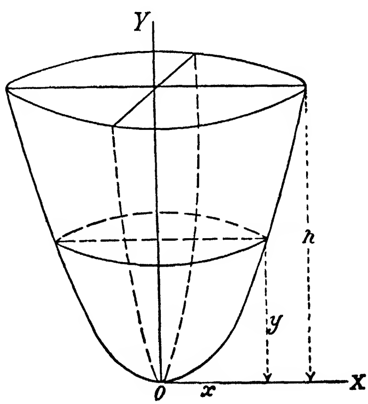
hence, $V = \frac{1}{6}(0 + 0 + 4A_0)\, 2a = \frac{4}{3}A_0 a = \frac{4}{3}\pi a^3$, where $A_0 = A_m = \pi a^2$.

Example 3. — In the case of the paraboloid of revolution, about the axis OY, of the curve $y = x^2$;

$$V = \frac{1}{6}(0 + A_h + 4A_m)h$$

$$= \frac{1}{6}\left(\pi h^2 + 4\frac{\pi h^2}{2}\right) = \frac{1}{2}\pi h^2.$$

Here $A_s = \pi y$ is a linear function of the distance y, for by (1), Art. 170, $a = 0$, $b = \pi$, $c = 0$; hence the formula holds.



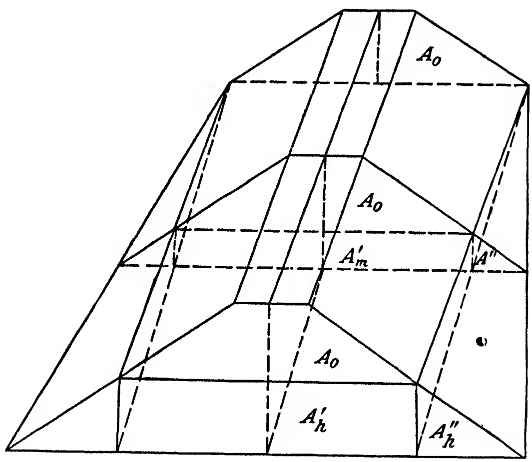
Example 4. — The prismoid shown in figure is composed of a prism, a wedge, and two pyramids. Let A_0 be the smaller end section, A_h the larger, and A_m the mid section.

$$V = A_0 h = \frac{h}{6}(A_0 + A_0 + 4\,A_0), \text{ for prism,}$$

$$V = A_h' \frac{h}{2} = \frac{h}{6}\left(0 + A_h' + 4\frac{A_h'}{2}\right), \text{ for wedge,}$$

$$V = A_h'' \frac{h}{3} = \frac{h}{6}\left(0 + A_h'' + \frac{4\,A_h''}{4}\right), \text{ for pyramid.}$$

The formula is seen to hold for the three forms of solids composing the prismoid.



As a practical case, let the figure represent a section of a railway embankment 100 feet in length.

A fill of 10 ft. with side slopes $1\frac{1}{2}$ to 1, makes $A_0 = 250$ sq. ft.

A fill of 20 ft. with side slopes $1\frac{1}{2}$ to 1, makes $A_h = 800$ sq. ft.

A fill of 15 ft. with side slopes $1\frac{1}{2}$ to 1, makes $4\,A_m = 1950$ sq. ft.

Hence, $V = \frac{100}{6}(250 + 800 + 1950) = 50{,}000$ cu. ft.

Here $V = h/2\,(A_0 + A_h) = \frac{100}{2}\,(250 + 800) = 52{,}500$ cu. ft., by average end areas.

$V = hA_m = 100 \times 487.5 = 48{,}750$ cu. ft., by mean area.

It is seen that the error of the approximation by the average end areas is twice that by mean area and of opposite sign. Since the errors vary as the square of the difference in dimensions of the two end areas, when the end areas are very different, the true prismoid formula should be used, but when the end areas are alike, or nearly so, the approximate formulas may give results as nearly exact as may be desired.

EXERCISE XXXV.

1. Get the volume of a frustum of a solid included between the planes $s = 0$ and $s = h$, when the area A_s of a parallel cross section is a cubic function, $as^3 + bs^2 + cs + d$, of the distance s from one of the bounding planes; first by direct integration using the frustum formula, then by the prismoid formula. Thus prove the statement at the beginning of Art. 171.

2. Show according to (2) Art. 171, as in the case of volumes, that the area under any curve $y = f(x)$, where $f(x)$ is any quadratic or cubic function of x, between $x = a$ and $x = b$, is

$$\frac{b-a}{6}(y_a + y_b + 4\,y_m), \tag{1}$$

where y_a, y_b, y_m represent the values of y at $x = a$, $x = b$, and $x = \frac{1}{2}(a + b)$.

3. Find, first by direct integration, and then by (1) of Ex. 2, the areas under each of the following curves.

(*a*) $y = x^2$, between $x = 0$ and $x = 2$.

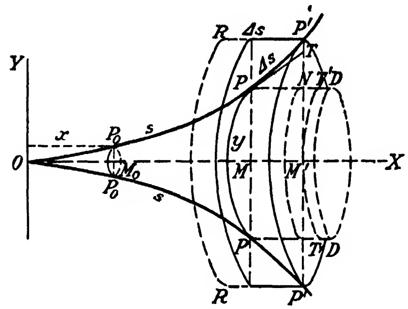
(*b*) $y = x^2 + 2\,x + 3$, between $x = 1$ and $x = 5$.

4. Show that, when (1) of Ex. 2 is used to get area under the curve $y = x^4$ between $x = 1$ and $x = 3$, the error is about 4.2 per cent.

5. Find the volume made by revolving the area between the curve $y = x^2$ and the x-axis about the x-axis, between $x = 0$ and $x = 2$. See Ex. 3, Art. 171. Find first by (1) of Art. 165; then by the prismoid formula show that the result by that formula is in error about 4.2 per cent.

Note. — The prismoid formula is not applicable for exact results, when A_s is given by a higher function than a cubic; in that case, it and the general formula (2), Art. 171, for $f(x)$, give approximations.

172. Surfaces and Solids of Revolution. — To get an expression for the area of a surface made by the revolution of a curve $y = f(x)$ about the axis OX, let $P_0 (x_0, y_0)$ be a fixed point and $P (x, y)$ a variable point on the curve OP_0P. Let $P_0P = s$, and $PP' = \Delta s$, and let PD and $P'R$ be drawn each parallel to OX and equal in length to Δs.



Let S denote the surface generated by the revolution of P_0P about the x-axis; then ΔS equals the surface generated by PP'. It is evident that

$$\text{surface } PD < \Delta S < \text{surface } P'R;$$

that is, $$2\pi y\,\Delta s < \Delta S < 2\pi (y + \Delta y)\,\Delta s;$$

dividing by Δs, $$2\pi y < \frac{\Delta S}{\Delta s} < 2\pi (y + \Delta y);$$

hence, $$\operatorname*{limit}_{\Delta s \doteq 0}\left[\frac{\Delta S}{\Delta s}\right] = \frac{dS}{ds} = 2\pi y, \text{ since } \Delta y \doteq 0, \text{ as } \Delta s \doteq 0;$$

$$\therefore \quad dS = 2\pi y\,ds \text{ or } S = 2\pi \int_0^s y\,ds. \quad \text{(See (3) Art. 169.)} \qquad (1)$$

Here $dS = 2\pi y\,ds$ may be represented by the lateral surface of a cylinder MPT', the circumference of whose base is $2\pi y$ and whose length is PT', drawn parallel to OX and equal to PT, which represents ds along the tangent at P. This is so, for this surface is what the change of S *would* be, if at P the change became uniform, ds being the uniform change of s as x increases uniformly from that point. The surface S may be considered as generated by the circumference of a circle of varying radius y and hence the point P moving on the curve according to the law expressed by its equation $y = f(x)$. Since

$$ds = (dx^2 + dy^2)^{\frac{1}{2}} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \text{ or } \left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{\frac{1}{2}} dy,$$

(1) becomes

$$S = 2\pi \int_{x_0}^{x} y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx, \tag{2}$$

or

$$S = 2\pi \int_{y_0} y\left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{\frac{1}{2}} dy. \tag{3}$$

Similarly, when the y-axis is the axis of revolution,

$$S = 2\pi \int_{x_0}^{x} x\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx, \tag{4}$$

or

$$S = 2\pi \int_{y_0}^{y} x\left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{\frac{1}{2}} dy. \tag{5}$$

These formulas may be derived as the limits of sums; thus,

$$S = \sum (\text{surfaces } \Delta s) = \lim_{\Delta s \doteq 0} \sum_0^s (\text{surfaces chord } \Delta s),$$

$$\left(\frac{\text{arc } \Delta s}{\text{chord } \Delta s} \doteq 1\right)$$

$$= \lim_{\Delta x \doteq 0} \sum_0^s 2\pi y \, \Delta s = \lim_{\Delta x \doteq 0} \sum_{x_0}^{x} 2\pi y\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]^{\frac{1}{2}} \Delta x$$

$$= 2\pi \int_{x_0}^{x} y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx. \tag{2}$$

The other forms may be derived in the same way, which is an abbreviation of a rigorous derivation.

In any particular example to which these formulas are applicable, use that form which involves the simpler integration.

For volumes of solids of revolution;

$$V = \pi \int_{x_0}^{x} y^2 \, dx = \pi \int_0^x (f(x))^2 \, dx, \text{ (See (4), Art. 169)} \tag{6}$$

when the revolution is about the x-axis; and

$$V = \pi \int_{y_0}^{y} x^2 \, dy = \pi \int_{y_0}^{y} (f(y))^2 \, dy, \tag{7}$$

when the y-axis is the axis of revolution.

A derivation as the limit of a sum has been given in Art. 165. In the figure of this Art. 172, if V is the volume made by the revolution of the area M_0P_0PM about OX, then dV is the volume of the cylinder MPN, whose base is πy^2 and whose length is $PN = dx$. This is so, for this volume is what the change of the volume V *would* be, if at P the change became uniform, as x increased uniformly by $\Delta x = dx$ from that point. As in the case of the surface S the volume V may be considered as being generated by a circle of varying radius y, the center of the moving circle always on the x-axis and the point P moving on the curve according to its equation.

By the method of limits, it is evident that, if $P'R = \Delta s$,

$$\text{volume } MM'P' > \Delta V > M'MP;$$

that is, $$\pi (y + \Delta y)^2 \Delta x > \Delta V > \pi y^2 \Delta x;$$

dividing by Δx, $$\pi (y + \Delta y)^2 > \frac{\Delta V}{\Delta x} > \pi y^2;$$

hence, $$\lim_{\Delta x \doteq 0} \left[\frac{\Delta V}{\Delta x}\right] = \frac{dV}{dx} = \pi y^2, \text{ since } \Delta y \doteq 0, \text{ as } \Delta x \doteq 0;$$

$$\therefore \quad dV = \pi y^2 \, dx \quad \text{or} \quad V = \pi \int_{x_0}^{x} y^2 \, dx. \tag{6}$$

If the revolution is made about a line $y = b$, then

$$V = \pi \int_{x_0}^{x} (y - b)^2 \, dx, \tag{8}$$

and when the revolution is about a line $x = a$, then

$$V = \pi \int_{y_0}^{y} (x - a)^2 \, dy. \tag{9}$$

Note. — It may be noted that the cone and the sphere of Art. 167 and the paraboloids of Art. 168 and Art. 171 are all solids of revolution, and hence the formulas of this Art. 172 are applicable to the determination of their surfaces and volumes.

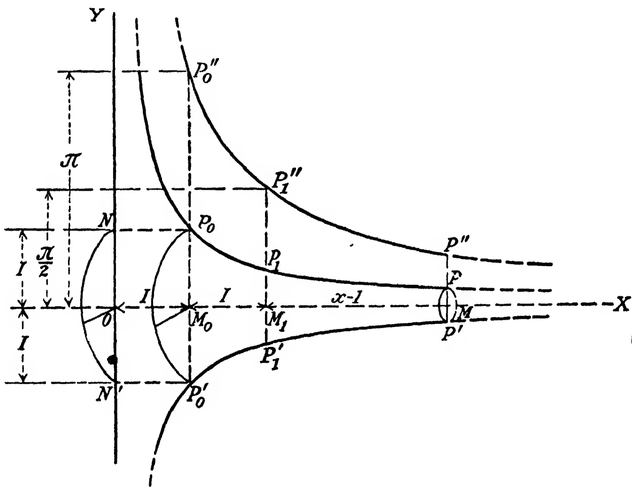
Example 1. — Find the volume generated by the revolution of the area of the equilateral hyperbola $xy = 1$ about OX.

$$V = \pi \int y^2\,dx = \pi \int_{x_0}^{x} \frac{1}{x^2}\,dx = -\pi \left[\frac{1}{x}\right]_{x_0}^{x} = \pi \left[\frac{1}{x_0} - \frac{1}{x}\right];$$

$$V = \pi \left[\frac{1}{x_0} - \frac{1}{x}\right]_{x=\infty}^{x_0=0} = \infty;$$

hence, the entire volume has no limit.

$$V = \pi \left[\frac{1}{x_0} - \frac{1}{x}\right]_{x=\infty}^{x_0=1} = \pi \text{ cubic units;}$$



hence the limit of the volume, from the section at $x_0 = OM_0 = 1$, extending indefinitely to the right, is the same as the volume of the cylinder generated by the revolution of NP_0, the abscissa of P_0, about OX. Thus, while the area under the curve $y = 1/x$, from the ordinate M_0P_0 at $x = 1$, indefinitely to the right, is unlimited (as shown in Ex. 11, Art. 135), the volume made by its revolution about OX has a definite limit. According to Art. 168, if the curve $y'' = \pi/x$

is drawn, any one of its ordinates in linear units will represent the volume of the solid extending indefinitely to the right of that ordinate; thus, in the figure the ordinate $M_0P_0'' = \pi$ represents the volume to the right of $P_0M_0P_0'$, and the ordinate $M_1P_1'' = \frac{1}{2}\pi$, the volume to the right of $P_1M_1P_1'$. In general, the ordinate MP'' at $x = OM$ represents the volume of the solid to the right of the section at any distance x from the origin, and it represents also the area under the curve $y = \pi/x^2$ to the right of the ordinate to that curve.

Example 2. — Find the volume to the left of the y-axis of the solid generated by the revolution of the exponential curve $y = e^x$ about the x-axis.

$$V = \pi\int_{y=0}^{y=1} y^2\,dx = \pi\int_{x=-\infty}^{x=0} e^{2x}\,dx = \frac{\pi}{2}e^{2x}\Big]_{-\infty}^{0} = \frac{\pi}{2}\text{ cubic units.}$$

(See Ex. 2, Art. 130, for figure.)

EXERCISE XXXVI.

In these examples, a *segment* of a solid of revolution means the portion included between two planes perpendicular to its axis, the solid or its segment being, in general, a *frustum;* and a zone means the *convex* surface of a segment.

1. Find the area of a zone of the paraboloid of revolution about the x-axis. $y^2 = 2px$, the plane curve. *Ans.* $\frac{2\pi}{3p}[(p^2+y^2)^{\frac{3}{2}} - (p^2+y_0^2)^{\frac{3}{2}}]$.

See Ex. 2, Art. 157, where $p = \frac{1}{8}$, $y_0 = 0$.

2. Find the area of a zone of the ellipsoid of revolution about the x-axis; that is, a zone of the prolate spheroid. Get entire surface.

$y^2 = \frac{b^2}{a^2}(a^2 - x^2) = (1 - e^2)(a^2 - x^2)$, where e is the eccentricity.

$$y\,ds = \frac{b}{a}\sqrt{a^2 - e^2x^2}\,dx.$$

$$\therefore\quad S = 2\pi\frac{b}{a}\int_x^x \sqrt{a^2 - e^2x^2}\,dx$$

$$= \pi\frac{b}{a}\left[x\sqrt{a^2 - e^2x^2} + \frac{a^2}{e}\sin^{-1}\frac{ex}{a}\right]_{x_0}^{x}.$$

The entire surface $= 2\pi b\,[b + (a/e)\sin^{-1}e]$.

The surface of a sphere $= \lim\limits_{e \doteq 0} 2\pi b\,[b + (a/e)\sin^{-1} e] = 2\pi a\,[a + a] = 4\pi a^2$,

since for circle $e = 0$, $\lim\limits_{e \doteq 0} \left[\dfrac{\sin^{-1} e}{e}\right] = \lim\limits_{\theta \doteq 0}\left[\dfrac{\theta}{\sin\theta}\right] = 1$; $a = b$.

3. Find the area of the surface generated by the revolution of the cycloid about its base.

Taking the parametric equations of the cycloid,

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta);$$

$$dx = a(1 - \cos\theta)\,d\theta, \quad dy = a\sin\theta\,d\theta;$$

$$ds = \sqrt{dx^2 + dy^2} = a\sqrt{2(1 - \cos\theta)}\,d\theta.$$

$$S = 2\pi\int y\,ds = 2\pi a^2\int_0^{2\pi}\sqrt{2(1-\cos\theta)^3}\,d\theta = 16\pi a^2\int \sin^3\left(\frac{\theta}{2}\right)d\left(\frac{\theta}{2}\right) = \frac{64}{3}\pi a^2.$$

4. Find the surface generated by revolving the catenary about the y-axis, from $x = 0$ to $x = a$. Also about the x-axis.

Here $\quad y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \quad ds = \dfrac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)dx.$

$$S = 2\pi\int x\,ds = \pi\int_0^a x\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)dx$$

$$= \pi\left[x \cdot a\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) - a\int_0^a\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)dx\right], \text{ by parts,}$$

$$= \pi\left[ax\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) - a^2\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)\right]_0^a = 2\pi a^2(1 - e^{-1}).$$

About x-axis: $\quad S = 2\pi\displaystyle\int y\,ds = 2\pi\int_0^a \frac{a}{4}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2 dx$

$$= \pi\left[\frac{a^2}{4}\left(e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}\right) + ax\right]_0^a = \frac{\pi a^2}{4}(e^2 - e^{-2} + 4).$$

5. Find the entire surface generated by revolving the hypocycloid about the x-axis. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is the equation of the curve.

$$S = 2\pi\int y\,ds = 2 \cdot 2\pi\int_0^a y\left(\frac{a}{y}\right)^{\frac{1}{3}}dy = 4\pi a^{\frac{1}{3}} \cdot \frac{3}{5}y^{\frac{5}{3}}\Big]_0^a = \frac{12}{5}\pi a^2.$$

6. Find the area of a zone of the surface generated by the tractrix revolving about the x-axis. (See Art. 150, Applied Calculus.)

$$S = 2\pi\int y\,ds = 2\pi\int_{y_0}^{y} y\left(-\frac{a\,dy}{y}\right) = 2\pi a\Big[-y\Big]_{y_0}^{y} = 2\pi a(y_0 - y).$$

7. A quadrant of a circle is revolved about a tangent at one extremity. Find the area of the curved surface generated.

$$S = 2\pi \int (a - x)\, ds = 2\pi \int_0^a (a - x)\left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}} dx,$$

when tangent to $x^2 + y^2 = a^2$ is perpendicular to x-axis,

$$= 2\pi \left[\int_0^a \frac{a^2\, dx}{\sqrt{a^2 - x^2}} - a \int_0^a \frac{x\, dx}{\sqrt{a^2 - x^2}}\right]$$

$$= 2\pi \left[a^2 \sin^{-1}\frac{x}{a} + a\sqrt{a^2 - x^2}\right]_0^a = \pi a^2 (\pi - 2).$$

8. Find the volume of a segment of the prolate spheroid, and the entire volume. Find the latter to be two-thirds the volume of the circumscribed cylinder of revolution.

Ans. $\dfrac{\pi b^2}{a^2}\left[a^2 (x - x_0) - \dfrac{1}{3}(x^3 - x_0^3)\right]$.

9. Find the volume of the oblate spheroid, that is, the ellipsoid of revolution about the minor axis which is on the y-axis.

Find the volume to be two-thirds of that of the circumscribed cylinder of revolution.

10. Find the volume of the paraboloid made by $x^2 = 2py$ about the y-axis. (Compare Ex. 3, Art. 171.)

Find the volume to be one-half that of the circumscribed cylinder of revolution.

11. Find that the volume of the solid generated by revolving an arch of the cycloid about its base is five-eighths of the circumscribed cylinder.

Here $$V = 2\pi \int_0^{2a} \frac{y^3\, dy}{\sqrt{2ay - y^2}} \quad \text{or} \quad V = \pi a^3 \int_0^{2\pi} (1 - \cos\theta)^3\, d\theta.$$

12. Find the volume generated by the catenary revolving about the x-axis, from $x = a$ to $x = -a$. Also find the volume by the area with the same arc revolving about the y-axis.

Here $$V = \pi \int_{-a}^{a} \frac{a^2}{4}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2 dx = \frac{\pi a^2}{4}\left[\frac{a}{2} e^{\frac{2x}{a}} + 2x - \frac{a}{2} e^{-\frac{2x}{a}}\right]_{-a}^{a}$$

$$= \frac{\pi a^3}{4}(e^2 + 4 - e^{-2}) = 8.83\, a^3.$$

And $$V = \pi \int_0^a x^2\, dy = \frac{\pi}{2} \int_0^a x^2 \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) dx.$$

Integrating by parts gives

$$V = \frac{\pi}{2}\left[ax^2\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) - 2a^2x\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) + 2a^2\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)\right]_0^a$$

$$= \frac{\pi a^3}{2}(e + 5e^{-1} - 4) = 0.878\,a^3.$$

13. Find the volume of the solid generated by the revolution of the tractrix about the x-axis.

$$V = \pi\int_0^\infty y^2\,dx = -\pi\int_a^0 \sqrt{a^2 - y^2}\,y\,dy = \frac{\pi a^3}{3}.$$

14. Find the volume generated by the revolution of the hypocycloid about the x-axis.

$$V = \pi\int y^2\,dx = \pi\int_{-a}^{a}(a^{\frac{2}{3}} - x^{\frac{2}{3}})^3\,dx = \frac{32}{105}\pi a^3.$$

15. Find the volume generated by the revolution about the y-axis of the equilateral hyperbola $xy = 1$, from $x = 0$ to $x = 1$.

$$V = \pi\int_{x=0}^{x=1} x^2\,dy = \pi\int_{y=1}^{y=\infty}\frac{dy}{y^2} = -\frac{\pi}{y}\Big]_1^\infty = \pi \text{ cubic units.}$$

(Compare Ex. 1, Art. 172.)

16. Find the volume of the segment of the solid generated by the revolution of the equilateral hyperbola $x^2 - y^2 = a^2$ about the x-axis, the altitude of the segment being a, measured from the vertex.

Ans. $\frac{4}{3}\pi a^3$.

17. Find the volume generated by revolving about either axis the part of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ intercepted by the axes.

Ans. $\frac{1}{15}\pi a^3$.

18. Find the volume of the solid generated by the quadrant of a circle revolved about a tangent at one extremity.

$$V = \pi\int_0^a (a - x)^2\,dy = \pi\int_0^a\left(a - \sqrt{a^2 - y^2}\right)^2 dy = \pi a^3\left(\frac{5}{3} - \frac{\pi}{2}\right).$$

19. Find the volume generated by the revolution of the cissoid $y^2 = \frac{x^3}{2a - x}$ about the x-axis, from the origin to $x = a$.

Ans. $\frac{8}{3}\pi a^3(3\log 2 - 2)$.

20. Find the volume generated by the revolution of the cissoid about its asymptote $x = 2a$.

Ans. $2\pi^2 a^3$.

CHAPTER V.

SUCCESSIVE INTEGRATION. MULTIPLE INTEGRALS. SURFACES AND VOLUMES.

173. Successive Integration. — As the inverse of successive differentiation there is successive integration. If a start is made with a function $y = f(x)$, considered as an nth derived function, a single integration gives another function, the integral; the integration of this function gives a second integral, and so on. The result of n integrations is the nth integral of the given function.

In Art. 140 on Integral Curves successive integration was indicated, and in Art. 141* the process was employed in application to beams. For successive integration with respect to a single independent variable, in general; let

$$f_1(x) = \int f(x)\,dx, \tag{1}$$

$$f_2(x) = \int f_1(x)\,dx, \tag{2}$$

$$f_3(x) = \int f_2(x)\,dx. \tag{3}$$

Since $$f_2(x) = \int [f_1(x)]\,dx,$$

it follows from (1) that

$$f_2(x) = \int \left[\int f(x)\,dx\right] dx; \tag{4}$$

and since $$f_3(x) = \int [f_2(x)]\,dx,$$

* Applied Calculus.

it follows from (4) that

$$f_3(x) = \int \left\{ \int \left[\int f(x)\,dx \right] dx \right\} dx. \qquad (5)$$

The integral in (4) is called a double integral and is written

$$\int\int f(x)\,dx^2.$$

Similarly, the integral in (5) is called a triple integral and is written

$$\int\int\int f(x)\,dx^3.$$

If an integral is evaluated by two or more successive integrations, it is called a *multiple integral.*

For example, to evaluate the multiple integral $\int\int\int e^x\,dx^3$;

$e^x + C_1$ is the first integral,
$e^x + C_1x + C_2$ is the second integral,
$e^x + \frac{C_1x^2}{2} + C_2x + C_3$ is the third integral.

Hence $$\int\int\int e^x\,dx^3 = e^x + \frac{C_1x^2}{2} + C_2x + C_3.$$

If limits are given for each successive integration, the integral is definite; if limits are not given, it is indefinite.

Example 1. — Given the acceleration $\frac{d^2s}{dt^2} = -g$ to find s.

This is Ex. 5 of Art. 115, and may be written thus:

$s = \int\int -g\,dt^2,$

$s = \int(-gt + v_0)\,dt,$ where v_0 is the constant of integration,

$s = -\frac{1}{2}gt^2 + v_0t + s_0,$ where s_0 is the constant of integration.

Example 2. — Determine the curve for every point of which the rate of change of the slope is 2.

Here $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dm}{dx} = 2;$$

$$\therefore \quad y = \int\int 2\,dx^2,$$

$$y = \int (2x + C_1)\,dx, \text{ where } 2x + C_1 \text{ is the first integral,}$$

$$y = x^2 + C_1x + C_2, \text{ the second integral.}$$

This is the equation of any parabola that has its axis parallel to the y-axis and drawn upwards, and its latus rectum equal to 1. All such parabolas may be gotten by giving all possible values to C_1 and C_2, the arbitrary constants of integration.

Example 3. — Determine the locus of the equation $\frac{d^2y}{dx^2} = 0$.

$$y = \int\int 0\,dx^2,$$

$$y = \int m\,dx, \text{ where } m \text{ is the constant of integration,}$$

$$y = mx + b, \text{ where } b \text{ is the constant of integration.}$$

The locus is the system of straight lines, the arbitrary constants m and b representing the slope and y-intercept, respectively.

Example 4. — In the theory of flexure of beams,

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[M + Rx - \frac{wx^2}{2}\right],$$

where $E, I, M, R,$ and w are constants. Get an expression for y and determine the constants of integration from the conditions, $y = 0$ when $x = 0$, and $y = 0$ when $x = l$.

$$y = \frac{1}{EI}\int\int\left[M + Rx - \frac{wx^2}{2}\right]dx^2,$$

$$y = \frac{1}{EI}\int\left(Mx + \frac{Rx^2}{2} - \frac{wx^3}{6} + C_1\right)dx,$$

$$y = \frac{1}{EI}\left[\frac{Mx^2}{2} + \frac{Rx^3}{6} - \frac{wx^4}{24} + C_1x + C_2\right]_{x=0} = 0; \quad \therefore \quad C_2 = 0,$$

$$0 = \frac{1}{EI}\left[\frac{Ml^2}{2} + \frac{Rl^3}{6} - \frac{wl^4}{24} + C_1 l\right], \text{ when } x = l;$$

$$\therefore \quad C_1 = \frac{1}{EI}\left[-\frac{Ml}{2} - \frac{Rl^2}{6} + \frac{wl^3}{24}\right],$$

$$y = \frac{1}{EI}\left[\frac{Mx^2}{2} + \frac{Rx^3}{6} - \frac{wx^4}{24} - \frac{Ml}{2} - \frac{Rl^2}{6} + \frac{wl^3}{24}\right].$$

Example 5. — Evaluate $\int_0^2 \int_1^3 \int_2^4 x^3\, dx^3$.

Letting I denote the integral and making the integrations *in order from right to left;*

$$I = \int_0^2 \int_1^3 \left[\frac{x^4}{4}\right]_2^4 dx^2 = 60 \int_0^2 \int_1^3 dx^2$$

$$= 60 \int_0^2 \left[x\right]_1^3 dx = 120 \int_0^2 dx = 240.$$

Example 6. — A point has an acceleration expressed by the equation $a_t = -r\omega^2 \sin \omega t$, where r and ω are constants. Get expressions for the velocity and the distance or space passed over.

$$\text{Here} \quad a_t = \frac{d^2 s}{dt^2} = -r\omega^2 \sin \omega t \text{ and } s = \int\int - r\omega^2 \sin \omega t\, dt^2,$$

$$\therefore \quad v = \frac{ds}{dt} = \int \frac{d^2 s}{dt^2} dt = -r\omega^2 \int \sin \omega t\, dt = r\omega \cos \omega t + C_1,$$

$$s = \int \frac{ds}{dt} dt = r\omega \int \cos \omega t\, dt + \int C_1\, dt,$$

$$s = r \sin \omega t + C_1 t + C_2,$$

which is the law of simple harmonic motion. (See Art. 73.)

EXERCISE XXXVII.

1. Evaluate $\int\int\int (x^2 - 1)\, dx^3$. *Ans.* $\frac{x^5}{60} - \frac{x^3}{6} + \frac{C_1 x^2}{2} + C_2 x + C_3$.

2. Evaluate $\int\int\int\int \frac{1}{x^4} dx^4$.

   Ans. $-\frac{1}{6} \log x + \frac{1}{6} x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4$.

3. Evaluate $\int\int\int\int \sin ax\, dx^4$.

Ans. $1/a^4 \sin ax + \frac{1}{6} C_1x^3 + \frac{1}{2} C_2x^2 + C_3x + C_4$.

4. Evaluate $\int_2^4\int_1^3\int_0^2 x^3\, dx^3$. *Ans.* 16.

5. Evaluate $\int_2^4\int_0^2\int_1^3 x^3\, dx^3$. *Ans.* 80.

6. Find the curve at each of whose points the rate of change of the slope is four times the abscissa, and which passes through the origin and the point (2, 4). *Ans.* $3y = 2x(x^2 - 1)$.

7. Evaluate $\int_0^{\frac{\pi}{2}}\int_\alpha^\beta\int_0^\pi \sin\theta\, d\theta^3$. *Ans.* $\pi(\beta - \alpha)$.

8. The differential equation of falling bodies is $\frac{d^2s}{dt^2} = -g$; show that $s = -\frac{gt^2}{2} + C_1t + C_2$; and find C_1 and C_2, if $s = 0$ and $v = 100$, when $t = 0$.

9. A point has an acceleration expressed by the equation $a_t = -r\omega^2 \cos\omega t$, where r and ω are constants. Find expressions for the velocity and the distance passed over. Find C_1 and C_2, if $s = r$ and $v = 0$, when $t = 0$.

Ans. $v = -r\omega \sin\omega t + C_1$; $s = r\cos\omega t + C_1t + C_2$

174. Successive Integration with Respect to Two or More Independent Variables. — In the preceding Article successive integration was of functions with respect to a single independent variable. Successive integration of functions of several independent variables are now to be considered. Suppose there is given a function $f(x, y, z)$ of three independent variables.

Let
$$f_1(x, y, z) = \int f(x, y, z)\, dz, \tag{1}$$
$$f_2(x, y, z) = \int f_1(x, y, z)\, dy, \tag{2}$$
$$f_3(x, y, z) = \int f_2(x, y, z)\, dx, \tag{3}$$

where in (1) the integration is with respect to z, that is, as if x and y were constants. Likewise in (2) it is with respect

to y, as if x and z were constants, and in (3) with respect to x, as if y and z were constants.

Equation (2), by substitution from (1), becomes

$$f_2(x, y, z) = \int \left[\int f(x, y, z)\, dz \right] dy; \tag{4}$$

and equation (3), by substitution from (4), becomes

$$f_3(x, y, z) = \int \left\{ \int \left[\int (x, y, z)\, dz \right] dy \right\} dx. \tag{5}$$

The integral in (5) is called a triple integral and is written

$$\int \int \int f(x, y, z)\, dx\, dy\, dz, \tag{6}$$

where the order of the integrations is from *right to left;* that is, the differential coefficient $f(x, y, z)$ is to be integrated with respect to z, that result to be integrated with respect to y, and finally the last result is to be integrated with respect to x.

Similarly, the double integral in (4) is written:

$$\int \int f(x, y, z)\, dy\, dz.$$

As to the *integration signs,* the first on the right is to be taken with the first differential on the right, which is dz in (6), the second sign from the right with the second differential from the right, and so on.

If when limits of integration are given, they are constant limits, the order of the integrations may be reversed without affecting the result, but when the definite integral has variable limits the order of the integrations can be changed only by new limits adapted to the new order. In practical problems the limits for one variable are often functions of one or more of the other variables.

175. The Constant of Integration. — The evaluation of an indefinite multiple integral differs from that of an indefinite single integral in the form of the constant of integration.

Thus $\int\int 4\,xy\,dx\,dy$ being given, to find a function u of x and y such that

$$\frac{\partial^2 u}{dx\,dy}dx\,dy = 4\,xy$$

is the problem. It is evident the operations represented by $\frac{\partial^2 u}{dx\,dy}\,dx\,dy$ must be reversed in order to get u.

That is, $$u = \int\int \frac{\partial^2 u}{dx\,dy}\,dx\,dy = \int\int 4\,xy\,dx\,dy, \tag{1}$$

which indicates two successive integrations, the first with respect to y, x and dx regarded as constants, and the second with respect to x, y being regarded as constant. Hence the first integration gives

$$\frac{\partial u}{dx} = 2\,xy^2 + \text{constant of integration.}$$

Since x was regarded as constant during the integration, the constant of integration may depend upon x, that is, it may be some function $\phi(x)$, or it may be simply C. This is so, since differentiating either $2\,xy^2 + C$, or $2\,xy^2 + \phi(x)$, with respect to y gives the same result, $4\,xy$. Hence,

$$\frac{\partial u}{dx} = 2\,xy^2 + \phi(x),$$

where $\phi(x)$ is an arbitrary function of x and may be a constant C.

Integrating this result, with y constant, gives

$$u = x^2y^2 + \int \phi(x)\,dx + F(y), \tag{2}$$

where, since y was regarded as constant during the integration, the integration constant is an arbitrary function of y and may be C with a constant value, possibly zero.

By referring to Art. 109, (2), it will be seen that if

$$u = x^3 + x^2y^2; \quad \frac{\partial^2 u}{dx\,dy}\,dx\,dy = 4\,xy\,dx\,dy;$$

that is, $\phi(x) = 3\,x^2$ and $F(y) = 0$, for that function u of (x, y).

The indefiniteness of the result in (2) is manifest, for

$$u = \int\int 4\,xy\,dx\,dy = x^2y^2,$$

if both constants of integration are zero, that is, $\phi(x) = 0$ and $F(y) = 0$. The indefiniteness is removed when limits for the variables are given, the integral being then a definite integral.

Example. —

$$\int_a^{2a}\int_0^x\int_y^x xyz\,dx\,dy\,dz = \int_a^{2a}\int_0^x xy\,dx\,dy\left[\frac{z^2}{2}\right]_y^x$$

$$= \int_a^{2a}\frac{x\,dx}{2}\int_0^x y\,(x^2 - y^2)\,dy$$

$$= \int_a^{2a}\frac{x^5}{8}\,dx = \frac{21}{16}\,a^6.$$

EXERCISE XXXVIII.

Evaluate the following integrals:

1. $\int\int x^2y\,dx\,dy.$ *Ans.* $\frac{1}{6}\,x^3y^2 + F(x) + f_1(y).$

2. $\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2y\,dx\,dy.$ *Ans.* $\frac{a^3b^2}{15}.$

3. $\int_0^a\int_0^{\frac{\pi}{2}} \rho^2\sin\theta\,d\rho\,d\theta = \frac{a^3}{3}.$

4. $\int_{\frac{b}{2}}^b\int_0^{\frac{\rho}{b}} \rho\,d\rho\,d\theta = \frac{7}{24}\,b^2.$

5. $\int_0^b\int_{y-b}^{2y} xy\,dy\,dx = \frac{11}{24}\,b^4.$

6. $\displaystyle\int_0^{\frac{\pi}{2}}\int_{2b\cos\theta}^{2a\cos\theta}\rho\,d\theta\,d\rho = \frac{\pi}{2}(a^2-b^2).$

7. $\displaystyle\int_a^b\int_\beta^\gamma \rho^2\sin\theta\,d\rho\,d\theta = \frac{b^3-a^3}{3}(\cos\beta-\cos\gamma).$

8. $\displaystyle\int_0^b\int_t^{10t}\sqrt{st-t^2}\,dt\,ds = 6\,b^3.$

9. $\displaystyle\int_2^3\int_1^2\int_2^5 xy^2\,dx\,dy\,dz = 17\tfrac{1}{2}.$

10. $\displaystyle\int_2^3\int_1^2\int_2^5 xy^2\,dz\,dy\,dx = 24\tfrac{1}{2}.$

11. $\displaystyle\int_2^5\int_1^2\int_2^3 xy^2\,dz\,dy\,dx = 17\tfrac{1}{2}.$

12. $\displaystyle\int_a^b\int_0^a\int_b^{2b} x^2y^2z\,dx\,dy\,dz = \tfrac{1}{6}\,a^3b^2(b^3-a^3).$

13. $\displaystyle\int_0^2\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz = \frac{e^8-3}{8} - \frac{3\,e^4}{4} + e^2.$

14. $\displaystyle\int_0^b\int_{h_1}^{h_2}\sqrt{2\,gy}\,dx\,dy = \tfrac{2}{3}\sqrt{2\,g}\,(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}})\,b.$

15. $\displaystyle\int_0^{\frac{\pi}{2}}\int_{a(1-\cos\theta)}^{a}\rho\,d\theta\,d\rho = \frac{a^2(8-\pi)}{8}.$

16. $\displaystyle\int_a^{2a}\int_v^{\frac{v^2}{a}}(w+2\,v)\,dv\,dw = \tfrac{148}{30}\,a^3.$

17. $\displaystyle\int_0^{2a}\int_0^x\int_{2y}^{3x} x^2\,yz\,dx\,dy\,dz = 32\,a^7.$

176. Plane Areas by Double Integration — *Rectangular coördinates.* — It has been shown in Art. 135, that the area between two curves $y = f(x)$ and $y = F(x)$ is given by

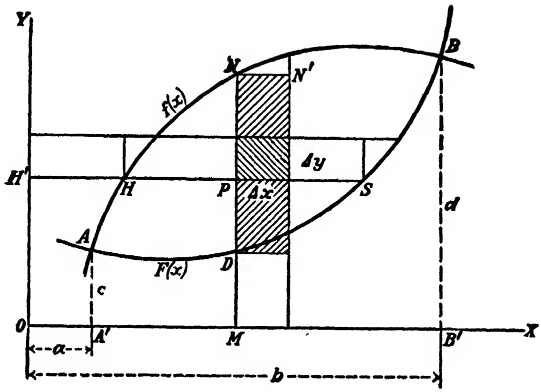
$$A = \int_{x_0}^{x_1}(f(x) - F(x))\,dx, \tag{1}$$

where the points of intersection are (x_0, y_0) and (x_1, y_1). The area is thus given not by a single integral but by the difference between two integrals, $\displaystyle\int_{x_0}^{x_1} f(x)\,dx - \int_{x_0}^{x_1} F(x)\,dx$. The result is gotten also by double integration, finding the limit of two sums. Let the element of area be $\Delta y\,\Delta x$, (x, y) being any point P of the area. If the elements are summed

up with respect to y, with the limits MD and MN, or $F(x)$ and $f(x)$, x being constant, the area of the strip DN' is gotten. If the strips are summed up with the limits a and b for x, then

$$\sum_{x=a}^{x=b}\left[\sum_{F(x)}^{f(x)} \Delta x\right]\Delta y = \sum_{a}^{b}\sum_{F(x)}^{f(x)} \Delta x\, \Delta y$$

is the expression for the sums. Taking the limits of the



sums, first as $\Delta y \doteq 0$ and then as $\Delta x \doteq 0$, the area $ADBN$ is given by the double integral

$$A = \int_a^b \int_{F(x)}^{f(x)} dx\, dy, \tag{2}$$

which integrated first with respect to y gives

$$A = \int_a^b (f(x) - F(x))\, dx. \tag{1}$$

If the elements are summed up in reverse order, first with respect to x with the limits $H'H$ and $H'S$, or $f^{-1}(y)$ and $F^{-1}(y)$, y being constant, and then with respect to y with limits c and d, there results

$$A = \int_c^d \int_{f^{-1}(y)}^{F^{-1}(y)} dy\, dx, \tag{3}$$

where $f^{-1}(y)$ and $F^{-1}(y)$ are the inverse functions of $f(x)$ and $F(x)$, respectively.

Integrating (3) the area $ADBN$ is gotten, as given by (2). Hence, in general,

$$A = \int\int dx\,dy \tag{4}$$

is the formula for area by double integration, the limits being taken so as to include the required area. The order of integration is indifferent provided the limits be adapted to the order taken.

Corollary. — $\partial_{xy}{}^2 A = dx\,dy$ and $\partial_{yx}{}^2 A = dy\,dx$.

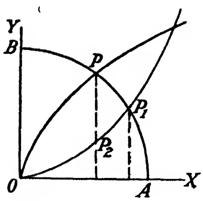
Example 1. — Find the area bounded by the parabolas $y^2 = 2px$ and $x^2 = 2py$.

The parabolas intersect at the points $(0, 0)$ and $(2p, 2p)$.

$$A = \int_0^{2p}\int_{\frac{x^2}{2p}}^{\sqrt{2px}} dx\,dy = \tfrac{4}{3}p^2, \text{ by formula (2).}$$

$$A = \int_0^{2p}\int_{\frac{y^2}{2p}}^{\sqrt{2py}} dy\,dx = \tfrac{4}{3}p^2, \text{ by formula (3).}$$

Example 2. — Find the area bounded by the circle $x^2 + y^2 = 12$, the parabola $y^2 = 4x$, and the parabola $x^2 = 4y$.



For the part OPP_2 the limits for x are 0 and 2, while for the part PP_1P_2, they are 2 and $\sqrt{8}$, the point P being $(2, \sqrt{8})$ and the point P_1 $(\sqrt{8}, 2)$. For both parts the lower limit for y is the ordinate of $x^2 = 4y$; for OPP_2 the upper limit for y is the ordinate of $y^2 = 4x$, and for PP_1P_2, that of $x^2 + y^2 = 12$.

$$A = OPP_1P_2 = OPP_2 + PP_1P_2 = \int_0^2\int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\,dy$$

$$+ \int_2^{\sqrt{8}}\int_{\frac{x^2}{4}}^{\sqrt{12-x^2}} dx\,dy = 3.92.$$

Example 3. — Find the area between the parabola $y^2 = ax$ and the circle $y^2 = 2\,ax - x^2$.

$$A = 2\int_0^a \int_{\sqrt{ax}}^{\sqrt{2\,ax - x^2}} dx\,dy$$

$$= 2\int_0^a \left(\sqrt{2\,ax - x^2} - \sqrt{ax}\right) dx = \frac{\pi a^2}{2} - \frac{4\,a^2}{3}.$$

Note. — It may be seen that in finding some areas there is no advantage in using double integration, as after the first integration with the limits substituted, the remaining integral is what might have been formed at first. There are, however, cases where double integration furnishes the only method of solution; hence the need for some practice in its application.

EXERCISE XXXIX.

1. Find the area between the circle $x^2 + y^2 = a^2$ and the line $y = a - x$.

$$A = \int_0^a \int_{a-x}^{\sqrt{a^2 - x^2}} dx\,dy = \frac{\pi - 2}{4}\,a^2.$$

2. Find by double integration the area between the parabolas $y^2 = 8\,x$ and $x^2 = 8\,y$.

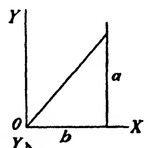
Ans. $21\frac{1}{3}$.

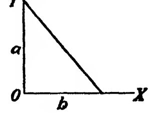
3. Find the area bounded by the circle $x^2 + y^2 = 25$, the parabola $y^3 = \frac{16}{3}\,x$, and the parabola $y = \frac{3}{16}\,x^2$.

Ans. 7.55.

4. Find by double integration the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5. Find the area of any right triangle, using double integration.

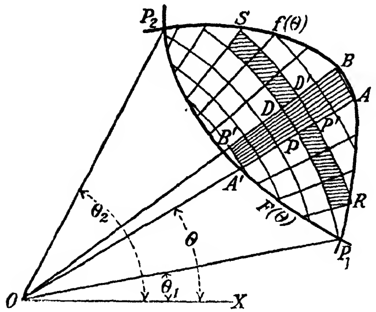




$$A = \int_0^b \int_0^{\frac{a}{b}x} dx\,dy = \int_0^b \frac{a}{b}\,x\,dx = \frac{a}{b}\,\frac{x^2}{2}\Big]_0^b = \frac{1}{2}\,ab.$$

$$A = \int^b \int^{-\frac{a}{b}x + a} dx\,dy = \int_0^b \left(-\frac{a}{b}\,x + a\right) dx = -\frac{a}{b}\,\frac{x^2}{2} + ax\Big]_0^b = \frac{1}{2}\,ab.$$

177. Plane Areas by Double Integration — *Polar coördinates.* — As has been shown in Art. 135(b), the area in polar coördinates of P_1OP_2, generated by the radius vector ρ as θ increases from θ_1 to θ_2 is given by



$$A = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\, d\theta. \qquad (1)$$

To find the area between two polar curves by double integration, let the element of area be $PDD'P'$, bounded by the two radii OP', OD', and the two circular arcs, concentric at 0.

Let the coördinates of P be (ρ, θ); then from geometry,

$$\text{sector } POD \;= \tfrac{1}{2}\rho^2\,\Delta\theta,$$
$$\text{sector } P'OD' = \tfrac{1}{2}(\rho + \Delta\rho)^2\,\Delta\theta.$$

Hence,
$$\Delta A = PDD'P' = \tfrac{1}{2}(\rho + \Delta\rho)^2\,\Delta\theta - \tfrac{1}{2}\rho^2\,\Delta\theta$$
$$= (\rho + \tfrac{1}{2}\Delta\rho)\,\Delta\theta\,\Delta\rho.$$

Keeping $\Delta\theta$ constant and summing the elements of area with respect to ρ gives an area $AA'B'B$, expressed by

$$\Delta\theta \cdot \lim_{\Delta\rho \doteq 0} \sum_{OA'}^{OA} (\rho + \tfrac{1}{2}\Delta\rho)\,\Delta\rho = \Delta\theta \int_{OA'}^{OA} \rho\, d\rho.$$

Making the summation now with respect to θ, the sum of the radial slices is gotten, and the limit of this sum is

$$A = \lim_{\Delta\theta \doteq 0} \sum_{\theta_1}^{\theta_2} \Delta\theta \cdot \int_{AO'}^{OA} \rho\, d\rho = \int_{\theta_1}^{\theta_2}\int_{OA'}^{OA} \rho\, d\rho\, d\theta.$$

Replacing OA' and OA by $F(\theta)$ and $f(\theta)$ respectively, the formula is

$$A = \int_{\theta_1}^{\theta_2}\int_{F(\theta)}^{f(\theta)} \rho\, d\theta\, d\rho. \qquad (2)$$

When $F(\theta) = 0$, the area P_1OP_2 between the curve $\rho = f(\theta)$

and the radii is (1), $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\, d\theta$, where (2) has been integrated as to ρ.

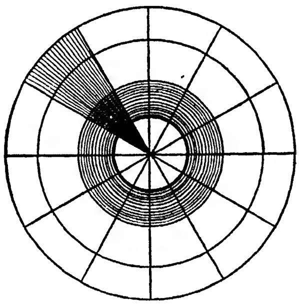
If the summing of the elements of area be made first with respect to θ, keeping $\Delta\rho$ constant, $RSDP$, a segment of a circular ring, is gotten. A second summation with respect to ρ gives the sum of such ring segments, the limit of which sum is the area A. The resulting formula is

$$A = \int_{\rho_1}^{\rho_2}\int_{F^{-1}(\rho)}^{f^{-1}(\rho)} \rho\, d\rho\, d\theta, \tag{3}$$

where $F^{-1}(\rho)$ and $f^{-1}(\rho)$ are the inverse functions of $F(\theta)$ and $f(\theta)$, respectively.

Corollary.— $\partial_{\theta\rho}{}^2A = \rho d\theta\, d\rho$ and $\partial_{\rho\theta}{}^2A = \rho\, d\rho\, d\theta$ are rectangles with sides $\rho\, d\theta$ and $d\rho$.

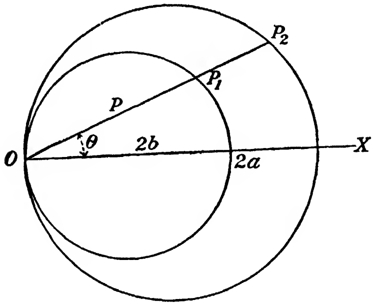
Example 1.— A simple case of the application of the formulas is in finding the area of the circle $\rho = a$.



(2) $$A = \int_0^{2\pi}\int_0^a \rho\, d\theta\, d\rho = \frac{1}{2}\int_0^{2\pi} \rho^2\, d\theta\Big]_0^a$$
$$= \frac{1}{2}a^2\theta\Big]_0^{2\pi} = \pi a^2.$$

(3) $$A = \int_0^a\int_0^{2\pi} \rho d\rho\, d\theta = \int_0^a \rho\, d\rho\, \theta\Big]_0^{2\pi}$$
$$= 2\pi\frac{\rho^2}{2}\Big]_0^a = \pi a^2.$$

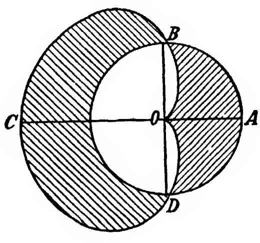
In (2) the sectors are summed, while in (3) the rings are summed. In this case of the circle it is to be noted that no *limit* need be invoked, since the *integral* is the *sum* in each case, the increments being the differentials, the variables all increasing uniformly.



Example 2. — Find the area between the two tangent circles $\rho = 2\,a\cos\theta$ and $\rho = 2\,b\cos\theta$, where $a > b$.

$$A = 2\int_0^{\frac{\pi}{2}}\int_{2\,b\cos\theta}^{2\,a\cos\theta}\rho\,d\theta\,d\rho = 4\,(a^2 - b^2)\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$
$$= \pi\,(a^2 - b^2).$$

Example 3. — Find the areas between the cardioid $\rho = 2\,a\,(1 - \cos\theta)$ and the circle $\rho = 2\,a$.



$$A = OBAD = 2\int_0^{\frac{\pi}{2}}\int_{2\,a\,(1-\cos\theta)}^{2\,a}\rho\,d\theta\,d\rho$$
$$= 4\,a^2\int_0^{\frac{\pi}{2}}[1 - (1 - \cos\theta)^2]\,d\theta$$
$$= 4\,a^2\int_0^{\frac{\pi}{2}}(2\cos\theta - \cos^2\theta)\,d\theta = 8\,a^2 - \pi a^2.$$

$$A = OBCD = 2\int_{\frac{\pi}{2}}^{\pi}\int_{2a}^{2a(1-\cos\theta)} \rho\, d\theta\, d\rho$$

$$= 4a^2 \int_{\frac{\pi}{2}}^{\pi} [(1-\cos\theta)^2 - 1]\, d\theta$$

$$= 4a^2 \int_{\frac{\pi}{2}}^{\pi} (-2\cos\theta + \cos^2\theta)\, d\theta = 8a^2 + \pi a^2.$$

EXERCISE XL.

1. Find by double integration the entire area of the cardioid $\rho = 2a(1-\cos\theta)$. *Ans.* $6\pi a^2$.

2. Find the area (1) between the first and the second spire of the spiral of Archimedes $\rho = a\theta$; (2) between any two consecutive spires; (3) the area described by the radius vector in one revolution from $\theta = 0$, and the area added by the nth revolution.

Ans. (1) $\frac{28}{3}\pi^3 a^2$; (2) $(n^2 + 2n + \frac{4}{3})\pi^3 a^2$; (3) $\frac{4}{3}\pi^3 a^2$, $\frac{4}{3}(n^3 - 1)\pi^3 a^2$.

3. Find by double integration the area of one loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$. *Ans.* $\frac{1}{2}a^2$.

4. Find by double integration the area between the circle $\rho = \cos\theta$ and one loop of the lemniscate $\rho^2 = \cos 2\theta$. Get the area between the circle and the line $\theta = \pi/4$ and then between that line, the lemniscate, and the circle. *Ans.* $\frac{\pi - 2}{4} = \frac{\pi - 2}{8} + \frac{\pi - 2}{8}$.

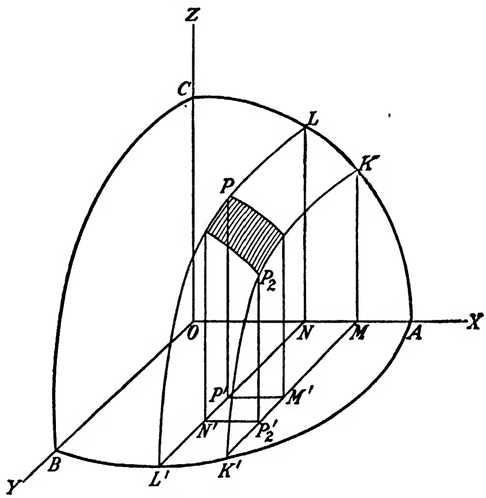
178. Area of any Surface by Double Integration. — Let the surface be given by an equation between the rectangular coördinates, x, y, z. Let the equation of the given surface be

$$z = f(x, y).$$

Passing two series of planes parallel, respectively, to XZ and YZ, will divide the given surface into elements. These planes will at the same time divide the plane XY into elementary rectangles, one of which is $P'P_2'$, the projection upon the plane XY of the corresponding element of the surface PP_2.

Let x, y, z be the coördinates of P and $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, those of P_2, x and y being independent; then $P'M' = \Delta x$ and $P'N' = \Delta y$. The planes which cut the element PP_2

from the surface will cut a parallelogram from the tangent plane at P, the projection of which on the plane XY is $P'P_2'$ $= \Delta x\ \Delta y$, the same as the projection of the element PP_2. The projection is the product of the area of the parallelogram



and the cosine of the angle made by the tangent plane with the plane XY; hence, denoting the angle by γ and the parallelogram cut from the tangent plane by PT,

$$\begin{aligned} \text{area } PT &= \text{area } P'P_2' \cdot \sec\gamma \\ &= \Delta x\, \Delta y \sec\gamma. \end{aligned}$$

As Δx and Δy approach zero, the point P_2 approaches the point P, and the areas PT and PP_2 approach equality; that is, the element of surface approaches coincidence with the parallelogram, a portion of the tangent plane at P; hence,

$$\text{area } PP_2 = \Delta_{xy}{}^2 S = \Delta x\, \Delta y \sec\gamma, \text{ approximately;}$$

that is,

$$\text{area } PP_2 = \Delta_{xy}{}^2 S \doteq \Delta x\, \Delta y \sec\gamma;\ \lim_{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}} \left[\frac{\Delta_{xy}{}^2 S}{\Delta x\, \Delta y}\right] = \sec\gamma;\ \binom{\text{Art.}}{21.}$$

$$\therefore\quad \partial_{xy}{}^2 S = \sec\gamma \cdot dx\, dy. \tag{1}$$

From Art. 103 (8), $\sec\gamma = \left[1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2\right]^{\frac{1}{2}}$; (See figure of Art. 101.)

hence from (1),

$$S = \int\int\left[1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2\right]^{\frac{1}{2}} dx\, dy, \tag{2}$$

the limits being so taken as to include the desired surface.

Let S denote that part of the surface $z = f(x, y)$, z being a one-valued function, which is included by the cylindrical surfaces $y = \phi_0(x)$, $y = \phi(x)$, and the planes $x = a$, $y = b$;

then $$S = \int_a^b \int_{\phi_0(x)}^{\phi(x)} \left[1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2\right]^{\frac{1}{2}} dx\, dy. \tag{2'}$$

In finding the area of the given surface a more convenient form of the equation of the surface may be either $x = f(y, z)$, or $y = f(z, x)$. The formula for the area will be then either

$$\int\int\left[1 + \left(\frac{\partial x}{dy}\right)^2 + \left(\frac{\partial x}{dz}\right)^2\right]^{\frac{1}{2}} dy\, dz, \tag{3}$$

or

$$\int\int\left[1 + \left(\frac{\partial y}{dx}\right)^2 + \left(\frac{\partial y}{dz}\right)^2\right]^{\frac{1}{2}} dx\, dz, \tag{4}$$

with the proper limits of integration.

In applying the formulas, the values of the partial derivatives are gotten from the equation of the surface the area of which is sought; hence, when there are two surfaces each of which intercepts a portion of the other, the partial derivatives in each case are taken from the equation of that surface whose partial area is being sought. This will be illustrated in the following examples.

Example 1. — To find the surface of the sphere whose equation is

$$x^2 + y^2 + z^2 = a^2.$$

Let $0 - ABC$ of the figure (Art. 178) be one-eighth of the sphere.

$$\frac{\partial z}{dx} = -\frac{x}{z}, \quad \frac{\partial z}{dy} = -\frac{y}{z};$$

$$1 + \left(\frac{\partial z}{dx}\right)^2 + \left(\frac{\partial z}{dy}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{a^2}{z^2} = \frac{a^2}{a^2 - x^2 - y^2}.$$

$$\text{Area} = S = 8\,ABC = 8\,a \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2 - x^2 - y^2}} \quad \text{by (2)}$$

$$= 8\,a \int_0^a \sin^{-1} \frac{y}{\sqrt{a^2 - x^2}}\Big]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\,\pi a \int_0^a dx = 4\,\pi a^2. \quad \left(\begin{array}{c}\text{Compare Ex. 2,}\\ \text{Exercise XXXVI.}\end{array}\right)$$

Here the integration was over the region OAB, the projection of the curved surface ABC on XY plane. The first integration with respect to y summed all the elements in a strip $LL'K'K$, y varying from zero to NL', that is, between limits 0 and $\sqrt{a^2 - x^2}$, the equation of the intersection of the surface with the XY plane being $x^2 + y^2 = a^2$. Integrating next with respect to x, the surface ABC is gotten by summing all the strips from $x = 0$ to $x = a$.

Example 2. — Find the area of the portion of the surface of a sphere which is intercepted by a right cylinder, one of whose edges passes through the center of the sphere, and the radius of whose base is half that of the sphere.

Note. — This is the celebrated Florentine enigma, proposed by Vincent Viviani as a challenge to the mathematicians of his time. (Williamson's Integral Calculus.)

Taking the origin at the center of the sphere, an element of the cylinder for the z-axis and a diameter of a right section of the cylinder for the x-axis, the equation of the sphere will be $x^2 + y^2 + z^2 = a^2$, and the equation of the cylinder,

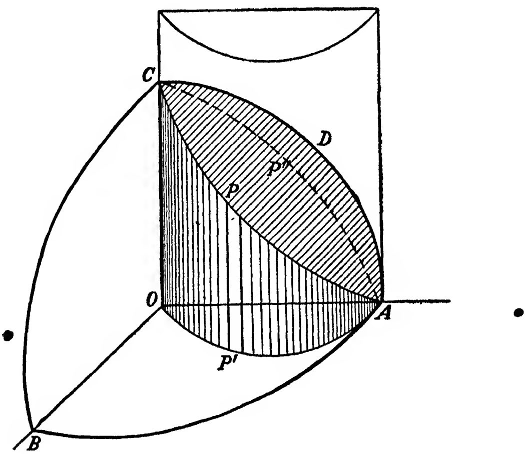
$$x^2 + y^2 = ax.$$

The area of $APCD$ is one-fourth of the area sought, and since this surface is a portion of the surface of the sphere, the partial derivatives $\frac{\partial z}{dx}, \frac{\partial z}{dy}$ must be taken from $x^2 + y^2 + z^2 = a^2$, giving, as in Ex. 1, using formula (2),

$$S = \int\int \frac{a\,dx\,dy}{\sqrt{a^2 - x^2 - y^2}}$$

to be integrated over the region $OP'A$. Hence,

$$\text{Area} = S = 4\int_0^a \int_0^{\sqrt{ax - x^2}} \frac{a\,dx\,dy}{\sqrt{a^2 - x^2 - y^2}} = (2\pi - 4)\,a^2.$$



The limits for y are from $x^2 + y^2 = ax$, the equation of the curve $OP'A$, the boundary of the projection of the surface $APCD$ on the XY plane.

Example 3. — Find the surface of the cylinder of Ex. 2, intercepted by the sphere.

The area of $APCOP'$ is one-fourth of the area sought, and since it is a part of the lateral surface of the cylinder $x^2 + y^2 = ax$, the partial derivatives in formula (2) must be

taken from this equation. But from this equation $\frac{\partial z}{dx} = \infty$, $\frac{\partial z}{dy} = \infty$, and formula (2) does not apply, which is, moreover, evident since the element of surface is $dx\,dz$ in the strip $P'P$, and the area of the surface $APCOP'$ cannot be found from its projection on the XY plane, for this projection is the arc $AP'O$. The projection is made on the XZ plane and formula (4) used.

The partial derivatives are found to be

$$\frac{\partial y}{dx} = \frac{a - 2x}{y}, \quad \frac{\partial y}{dz} = 0.$$

Since P is on the sphere,

$$\overline{P'P}^2 = z^2 = a^2 - (x^2 + y^2) = a^2 - ax,$$

since P is on the cylinder. Hence,

$$\text{Area} = S = 4\int_0^a \int_0^{\sqrt{a^2-ax}} \left[1 + \left(\frac{a - 2x}{2y}\right)^2\right]^{\frac{1}{2}} dx\,dz$$

$$= 2a\int_0^a \int_0^{\sqrt{a^2-ax}} \frac{dx\,dz}{\sqrt{ax - x^2}} = 2a\int_0^a \frac{\sqrt{a^2 - ax}}{\sqrt{ax - x^2}}dx$$

$$= 2a\int_0^a \sqrt{\frac{a}{x}}\,dx = 4a^2.$$

Here the integration is over the region $OAP''C$, $AP''C$ being the projection of APC on XZ plane. The first integration sums up the elements of surface in the strip $P'P$ and the next integration sums up the strips from $x = 0$ to $x = a$.

By eliminating y from $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = ax$, $z^2 = a^2 - ax$ (as found above), which is the equation of $AP''C$, from which the limits of z are taken.

EXERCISE XLI.

Find by double integration the areas of the surfaces given in the following examples:

1. The zone of the sphere, $x^2 + y^2 + z^2 = r^2$, included between the planes $x = a$ and $x = b$. *Ans.* $2\pi r\,(b - a)$.

2. The surface of the right cylinder $x^2 + z^2 = a^2$ intercepted by the right cylinder $x^2 + y^2 = a^2$. Compare Ex. 8, Art. 169. *Ans.* $8\,a^2$.

3. The part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, in the first octant, intercepted by the coördinate planes. *Ans.* $\frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$.

4. The surface of the cylinder $x^2 + y^2 = a^2$, included between the plane $z = mx$ and the XY plane. Find by both formula (3) and formula (4), and show why formula (2) does not apply.

Ans. $4\,ma^2$.

5. The surface of the paraboloid of revolution $y^2 + z^2 = 4\,ax$, intercepted by the parabolic cylinder $y^2 = ax$ and the plane $x = 3\,a$.

$$S = 4\int_0^{3a}\int_0^{\sqrt{ax}}\left[\frac{4\,ax + 4\,a^2}{4\,ax - y^2}\right]^{\frac{1}{2}} dx\,dy = \frac{56}{9}\pi a^2.$$

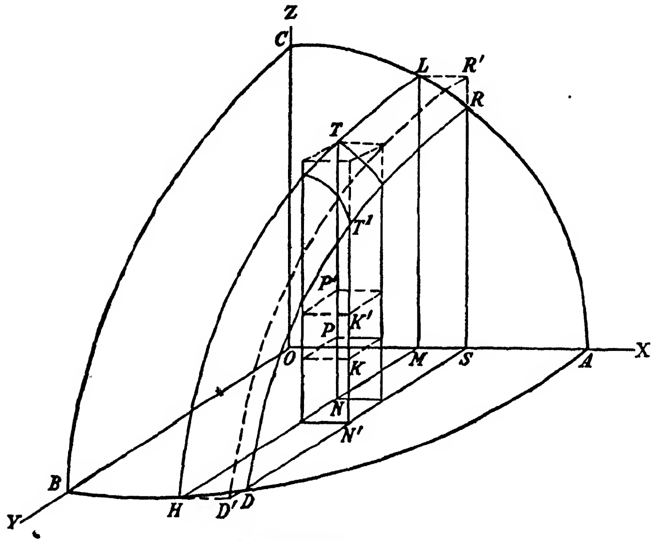
6. The surface of the cylinder of Ex. 5, intercepted by the paraboloid of revolution and the given plane.

$$\textit{Ans.}\quad S = 4\int_0^{3a}\int_0^{\sqrt{3ax}}\frac{[4\,y^2 + a^2]^{\frac{1}{2}}}{2\,y}\,dx\,dz = 2\sqrt{3}\int_0^{3a}(4\,ax + a^2)^{\frac{1}{2}}\,dx$$
$$= (13\sqrt{13} - 1)\frac{a^2}{\sqrt{3}}.$$

179. Volumes by Triple Integration — *Rectangular Coördinates.* — Let the volume be that of a solid bounded by the coördinate planes and any surface given by an equation between the coördinates x, y, and z.

Let P be any point (x, y, z) within the solid $O - ABC$, the surface being given by $z = f(x, y)$, where $f(x, y)$ is a continuous function. Let K' be the point $(x + \Delta x,\ y + \Delta y,\ z + \Delta z)$, the diagonally opposite corner of the rectangular parallelopiped formed by passing planes through P and K', the planes being parallel to the coördinate planes. Let more planes be passed. Taking first the sum of the elementary parallelopipeds whose edges lie along the line NT, the limit of this sum, as Δz is made to approach zero, is the volume of the prism whose base is $\Delta x\,\Delta y$ and whose altitude is NT, x, y, Δx, and Δy remaining constant during the summation. Next with x and Δx constant, sum the prisms between the planes MHL and SDR. The limit of this sum as Δy is made to approach zero is the volume of the cylindrical slice

$LR'D'HMS$. Finally, when taking the sum of the slices parallel to the YZ plane, as Δx approaches zero, the volume of the cylindrical slice approaches that of the actual slice $LRDHMS$; hence, the limit of the sum of the slices, as Δx approaches zero, is the volume of the solid.



Hence, $$V = \int_0^{OA}\int_0^{MH}\int_0^{NT} dx\,dy\,dz,$$

where V is the volume of $O - ABC$. Let V denote the volume bounded by the curved surfaces $z = f_0(x, y)$, $z = f(x, y)$; the cylindrical surfaces $y = \phi_0(x)$, $y = \phi(x)$; and the planes $x = a$, $x = b$; then

$$V = \int_a^b \int_{\phi_0(x)}^{\phi(x)} \int_{f_0(x,y)}^{f(x,y)} dx\,dy\,dz. \tag{1}$$

Corollary. — $\partial_{xyz}{}^3 V = dx\,dy\,dz$, $\partial_{yzx}{}^3 V = dy\,dz\,dx$, . . .

If z is expressed in terms of x and y, and $f_0(x, y) = 0$;

$$V = \int_a^b \int_{\phi_0(x)}^{\phi(x)} z\,dx\,dy.$$

Note. — The formula, $V = \int\int\int dx\, dy\, dz$, may be derived from the figure by the definition of differentials.

Thus, the variables x, y, z, being independent, dx, dy, dz may be taken as finite constants, the parallelopiped PK' being $dx\, dy\, dz$. When x and y are regarded constant, PK' is the differential of the prism NK. Hence, integrating $dx\, dy\, dz$ between the limits $z = 0$ and $z = NT$ gives the prism $\overline{NT}\, dx\, dy$, which is the differential of the solid $MSR'L - T$. Integrating $\overline{NT}\, dx\, dy$ between the limits $y = 0$ and $y = MH$ gives the cylinder $MLH - D'$, or $MLH\, dx$, which is the differential of the solid $OBC - M$. Integrating $MLH\, dx$ between the limits $x = 0$ and $x = OA$ gives the volume $OBC - A$, or V. Hence, $V = \int\int\int dx\, dy\, dz$, the limits being so chosen as to include the volume sought.

Example. — Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The entire volume is eight times that in the first octant, where the limits are:

$$z = 0, \quad z = c\sqrt{1 - x^2/a^2 + y^2/b^2};$$
$$y = 0, \quad y = b\sqrt{1 - x^2/a^2};$$
$$x = 0, \quad x = a;$$

$$\therefore \quad V = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dx\, dy\, dz = \frac{4\,\pi abc}{3}.$$

Corollary. — For sphere, $a = b = c$; $\quad \therefore \quad V = \dfrac{4\,\pi a^3}{3}.$

Also,
$$V = \int_0^b \int_0^{a\sqrt{1-y^2/b^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dy\, dx\, dz$$
$$= \int_0^c \int_0^{a\sqrt{1-z^2/c^2}} \int_0^{b\sqrt{1-x^2/a^2-z^2/c^2}} dz\, dx\, dy.$$

EXERCISE XLII.

Find by triple integration the volumes required:

1. The tetrahedron bounded by the coördinate planes and by the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad Ans.\ \frac{abc}{6}.$$

See Ex. 3, Exercise XLI, for the surface of the plane.

2. The volume bounded by the cylinder $x^2+y^2=a^2$ and the planes $z=0$ and $z=mx$. *Ans.* $\frac{4\,ma^3}{3}$.

3. A cylindrical vessel with a height of 12 inches and a base diameter of 8 inches is tipped and the contained liquid is poured out until the surface of the remaining liquid coincides with a diameter of the base. Find the volume remaining in the vessel. *Ans.* 128 cu. in.

Note that the volume is one-half that given by Ex. 2, above.

4. The volume included between the paraboloid of revolution $y^2+z^2=4\,ax$, the parabolic cylinder $y^2=ax$ and the plane $x=3\,a$. See Exs. 5 and 6, Exercise XLI, for the surfaces.

$$Ans.\ V=4\int_0^{3a}\int_0^{\sqrt{ax}}\int_0^{(4\,ax-y^2)^{\frac{1}{2}}}dx\,dy\,dz=\left(6\,\pi+9\sqrt{3}\right)a^3.$$

5. The volume included between the paraboloid of revolution $x^2+y^2=az$, the cylinder $x^2+y^2=2\,ax$, and the XY plane.

$$Ans.\ V=2\int_0^{2a}\int_0^{\sqrt{2\,ax-x^2}}\int_0^{\frac{x^2+y^2}{a}}dx\,dy\,dz=\tfrac{3}{2}\,\pi a^3.$$

6. The entire volume bounded by the surface

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1.$$

$$Ans.\ V=\int_0^{a}\int_0^{b\left(1-\sqrt{x/a}\right)^2}\int_0^{c\left(1-\sqrt{x/a}-\sqrt{y/b}\right)^2}dx\,dy\,dz=\frac{abc}{90}.$$

7. The entire volume bounded by the surface

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}. \qquad Ans.\ \frac{4\,\pi a^3}{35}.$$

8. The volume of the part of the cylinder intercepted by the sphere. The radius of the sphere is a and it has its center on the surface of a right cylinder, the radius of whose base is $a/2$. See Exs. 2 and 3, Art. 178.

$$Ans.\ V=4\int_0^{a}\int_0^{\sqrt{ax-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}dx\,dy\,dz=\tfrac{2}{3}\left(\pi-\tfrac{4}{3}\right)a^3.$$

180. Solids of Revolution by Double Integration. — In the figure of Art. 176, where $P(x, y)$ is any point in the area $ADBN$, x and y being independent, $\Delta x \, \Delta y$ is the element of area. Conceive the area $ADBN$ to revolve through θ radians about OX as an axis; then

$$\theta y \cdot \Delta x \, \Delta y < \Delta_{xy}{}^2 V < \theta\,(y + \Delta y) \cdot \Delta x \, \Delta y;$$

$$\therefore \quad \theta y < \frac{\Delta_{xy}{}^2 V}{\Delta x \, \Delta y} < \theta\,(y + \Delta y);$$

hence,
$$\lim_{\Delta x,\, \Delta y \doteq 0} \left[\frac{\Delta_{xy}{}^2 V}{\Delta x \, \Delta y}\right] = \frac{\partial^2 V}{dx\, dy} = \theta y;$$

$$\therefore \quad \partial^2 V = \theta y \, dx \, dy;$$

$$\therefore \quad V = \theta \int_{x_0}^{x_1} \int_{F(x)}^{f(x)} y \, dx \, dy. \tag{1}$$

Similarly, about OY,

$$V = \theta \int_{y_0}^{y_1} \int_{F^{-1}(y)}^{f^{-1}(y)} x \, dy \, dx. \tag{2}$$

Putting $\theta = 2\pi$, the formulas give the volumes generated by a complete revolution of the area.

Corollary. — If the axis of revolution cuts the area, (1) or (2) will give the difference between the volumes generated by the two parts. Hence $V = 0$, when these two parts generate equal volumes. Integrating (1) first with respect to y, and (2) first with respect to x, the upper limits being the variables y or x and the lower limits zero, gives

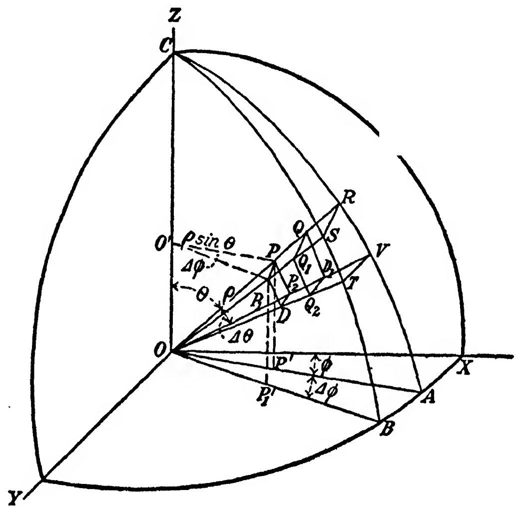
$$V = \pi \int_{x_0}^{x_1} y^2 \, dx \tag{1'}$$

and
$$V = \pi \int_{y_0}^{y_1} x^2 \, dy, \tag{2'}$$

the formulas for solids of revolution, single integration.

181. Volumes by Triple Integration — *Polar Coördinates.* — Let the point $P(\rho, \theta, \phi)$ be any point within a portion of a solid bounded by a surface and the rectangular planes. As usual, ρ is the distance OP from the pole at the origin, θ is the angle ZOP which OP makes with the z-axis,

and ϕ is the angle XOP' which the projection of OP on the XY plane makes with the x-axis. Let the solid be divided into elementary volumes like PDD_1Q_1 by the following means.



(1) Through the z-axis pass a series of consecutive planes, dividing the solid into wedge-shaped slices such as $COAB$.

(2) Round the z-axis describe a series of right cones with their vertices at O, thus dividing each slice into elementary pyramids like $O - RSTV$.

(3) With O as a center describe a series of consecutive spheres. Thus the solid is divided into elementary solids like PDD_1Q_1, whose volume is given approximately by the product of three of its edges, PP_1, PP_2, and PQ.

Let edge $PQ = \Delta\rho$, angle $POP_2 = \Delta\theta$, angle AOB = angle $PO'P_1 = \Delta\phi$; then edge $PP_1 = \rho \sin\theta\, \Delta\phi$, and edge $PP_2 = \rho\, \Delta\theta$.

Hence, the volume of the elementary solid is given approximately by $\rho^2 \sin\theta\, \Delta\theta\, \Delta\phi\, \Delta\rho$. It can be expressed

exactly but the additional terms vanish when the three increments are made to approach zero. Therefore, the volume of the solid is given by the limit of

$$\sum_{\Delta\theta \doteq 0} \sum_{\Delta\phi \doteq 0} \sum_{\Delta\rho \doteq 0} \rho^2 \sin\theta \, \Delta\theta \, \Delta\phi \, \Delta\rho;$$

$$\therefore \quad V = \int\int\int \rho^2 \sin\theta \, d\theta \, d\phi \, d\rho, \tag{1}$$

each integral to be taken between the limits required to find the volume sought. The summation can be made in any order so long as the volume is continuous.

For a solid of revolution with the z-axis as the axis of revolution, the formula (1) for the volume becomes

$$V = 2\pi \int\int \rho^2 \sin\theta \, d\theta \, d\rho, \tag{2}$$

since the limits for ϕ are then evidently 0 and 2π. The limits for ρ and θ are then the same as are used in getting the area of the plane figure revolved.

Corollary. — $\partial_{\rho\theta\phi}{}^3 V = \rho^2 \sin\theta \, d\theta \, d\phi \, d\rho$ is an elementary rectangular parallelopiped; and $\partial_{\rho\theta}{}^2 V = 2\pi\rho^2 \sin\theta \, d\theta \, d\rho$ is a circular ring with rectangular section.

Example 1. — Find the volume of a sphere of radius a, using polar coördinates, pole at end of a diameter.

By formula (1); or by (2), if the volume is considered as generated by revolving the semicircle about the initial line, the line from which θ is measured,

$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \rho^2 \sin\theta \, d\theta \, d\rho \\ &= \frac{16\pi a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta \, d\theta \\ &= \frac{16\pi a^3}{3} \left[-\frac{\cos^4\theta}{4} \right]_0^{\frac{\pi}{2}} = \frac{4}{3}\pi a^3. \end{aligned}$$

Example 2. — Find the volume generated by revolving the cardioid, $\rho = 2a(1 - \cos\theta)$ about the initial line.

$$V = 2\pi \int_0^{\pi} \int_0^{2a(1-\cos\theta)} \rho^2 \sin\theta \, d\theta \, d\rho$$

$$= \frac{16\pi a^3}{3} \int_0^{\pi} (1 - \cos\theta)^3 \sin\theta \, d\theta = \frac{64}{3}\pi a^3.$$

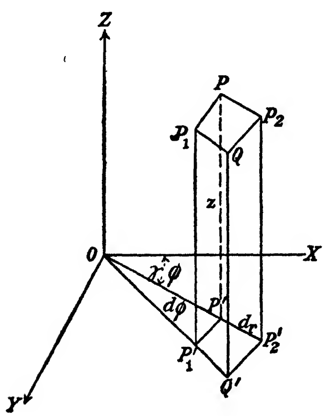
Example 3. — Find the volume made by revolving the lemniscate $\rho^2 = a^2 \cos 2\theta$ about the initial line.

$$V = 2\pi \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \rho^2 \sin\theta \, d\theta \, d\rho$$

$$= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \sin\theta \, d\theta = \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} (2\cos^2\theta - 1)^{\frac{3}{2}} \sin\theta \, d\theta$$

$$\doteq \pi a^3 \left(\frac{\log(\sqrt{2} + 1)}{2\sqrt{2}} - \frac{1}{6} \right).$$

182. Volumes by Double Integration — *Cylindrical Coördinates.* — In finding the volume of some solids the integration is performed more readily with the use of cylindrical coördinates.



In this system of coördinates the position of a point is given by the cylindrical coördinates (r, ϕ, z), where (r, ϕ) are the polar coördinates of the projection $(x, y, 0)$, on the XY plane, of the point (x, y, z).

It is evident that the equations of transformation from rectangular to cylindrical coördinates are:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z; \tag{1}$$

and those from cylindrical to rectangular,

$$r = \sqrt{x^2 + y^2}, \quad \phi = \cos^{-1}\frac{x}{r} = \sin^{-1}\frac{y}{r} = \tan^{-1}\frac{y}{x}, \quad z = z. \qquad (2)$$

To derive a formula for volume the differential element of area in the XY plane may be taken as the rectangular base of an elementary right prism with altitude z, the base of the actual prisms into which the solid may be divided being bounded by lines two only of which are right lines, the other two being circular arcs, and the altitude of possibly only one edge being z, since the surface of the solid may be curved, or not parallel to the XY plane, even when plane.

The expression for the volume of the solid is

$$V = \int\int zr\,d\phi\,dr, \qquad (1)$$

where z must be expressed in terms of r or ϕ in order to effect the integration, and where the limits are to be such as will give the volume sought.

Corollary. — $\partial_{r\phi}{}^2V = zr\,d\phi\,dr$ is a right prism with rectangular base. The double integral in (1) is the limit of the sum of the elementary solids into which the given solid is conceived to be divided, or it is to be considered simply as the anti-differential of a second partial differential, when the differentials are taken as finite constants. Either way of regarding the differentials leads to the same result.

Example 1. — To find the volume of a sphere of radius a. Taking the pole at the center of the sphere, by (1),

$$V = 2\int_0^{2\pi}\int_0^{a}\sqrt{a^2 - r^2}\,r\,d\phi\,dr$$

$$= \tfrac{2}{3}a^3\int_0^{2\pi} d\phi = \tfrac{4}{3}\pi a^3.$$

Example 2. — A cylindrical core with b as the radius of a right section is cut from a sphere of radius a. Find the

volume of the remaining portion of the sphere, when $b < a$ and the axis of the core includes a diameter of the sphere.

$$V = 2\int_0^{2\pi}\int_b^a \sqrt{a^2 - r^2}\, r\, d\phi\, dr$$

$$= 2\int_0^{2\pi} d\phi \left[-\tfrac{1}{3}(a^2 - r^2)^{\frac{3}{2}}\right]_b^a$$

$$= \tfrac{4}{3}\pi (a^2 - b^2)^{\frac{3}{2}};$$

$$\therefore \quad \text{Vol. of core} = \frac{4\pi}{3}\left(a^3 - (a^2 - b^2)^{\frac{3}{2}}\right).$$

Example 3. — Find the volume in first octant cut from a right cylinder, with its base of radius r_1 on XY plane and axis the z-axis, by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Here

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right) = c\left(1 - \frac{r}{a}\cos\phi - \frac{r}{b}\sin\phi\right).$$

$$V = c\int_0^{\frac{\pi}{2}}\int_0^{r_1}\left(1 - \frac{r}{a}\cos\phi - \frac{r}{b}\sin\phi\right) r\, d\phi\, dr$$

$$= c\int_0^{\frac{\pi}{2}}\left[\frac{r_1^2}{2} - \frac{r_1^3}{3}\left(\frac{\cos\phi}{a} + \frac{\sin\phi}{b}\right)\right] d\phi = r_1^2 c\left[\frac{\pi}{4} - \frac{r_1}{3}\left(\frac{1}{a} + \frac{1}{b}\right)\right].$$

Example 4. — Find the volume of Ex. 8, Exercise XLII, by using formula (1), which will give the volume sought more easily than by $\int\int\int dx\, dy\, dz$.

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\phi}\sqrt{a^2 - r^2}\, r\, d\phi\, dr = \tfrac{2}{3}\left(\pi - \tfrac{4}{3}\right) a^3.$$

Example 5. — Find the volume of the segment of the right cylinder which has its base a loop of the lemniscate $r^2 = a^2\cos 2\phi$ in the XY plane and its upper surface a plane which intersects the XY plane in the y-axis at an angle of $45°$.

Here $\qquad z = x = r\cos\phi;$

$$\therefore \quad V = 2\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\phi}} r\cos\phi r\,d\phi\,dr$$

$$= \frac{2a^3}{3}\int_0^{\frac{\pi}{4}} \cos^{\frac{3}{2}} 2\phi \cos\phi\,d\phi$$

$$= \frac{2a^3}{3}\int_0^{\frac{\pi}{4}} (1 - 2\sin^2\phi)^{\frac{3}{2}} \cos\phi\,d\phi$$

$$= \frac{\pi\sqrt{2}\,a^3}{16}.$$

183. Mass. *Mean Density.* — As stated in Art. 166, the mass of a body, being defined as the product of density and volume, when the density * varies continuously,

$$m = \underset{\Delta V \doteq 0}{\text{limit}} \sum \gamma\,\Delta V = \int \gamma\,dV, \tag{1}$$

which becomes $m = \iiint \gamma\,dx\,dy\,dz,$ (2)

or

$$m = \iiint \gamma\,\rho^2 \sin\theta\,d\theta\,d\phi\,d\rho, \tag{3}$$

according as rectangular or polar elements of volume are used. In these expressions γ denotes the varying density at the different points within the body. The mean density of the body, denoted by $\bar{\gamma}$, is given by the equation

$$\bar{\gamma} = \frac{m}{V} = \frac{\int \gamma\,dV}{V}. \tag{4}$$

When the mass is considered as distributed continuously over a surface, the element of volume dV is replaced by $dA = dx\,dy$ or $\rho\,d\theta\,d\rho$; and when the mass is considered as

* Density will now be denoted by γ, instead of ρ used in Art. 166.

distributed along a line, straight or curved, dV is replaced by ds, the element of length.

When the integral is considered as an anti-differential the elements are expressed directly in terms of the finite differentials; when, however, the integral is considered as the limit of a sum or sums, the elements are expressed in terms of the infinitesimal increments, the differentials appearing under the integral sign.

Example 1. — Find the mean density of a sphere in which the density varies as the square of the distance from the center.

Here the distance ρ of a volume element determining its density, the polar element should be used.

Taking the density at a distance ρ from the center as $k\rho^2$, k being a constant, and the volume element as $\rho^2 \sin\theta\, \Delta\theta\, \Delta\phi\, \Delta\rho$, from (4),

$$\bar{\gamma} = \frac{\int_0^{\pi}\int_0^{2\pi}\int_0^{a} k\rho^4 \sin\theta\, d\theta\, d\phi\, d\rho}{\frac{4}{3}\pi a^3} = \frac{3}{5}ka^2.$$

Again, since the density is the same for all points at the same distance ρ from the center, taking for the volume element a spherical shell of thickness $\Delta\rho$, $\Delta V = 4\pi\rho^2\Delta\rho$, whence

$$\bar{\gamma} = \frac{4\pi\int_0^{a} k\rho^4\, d\rho}{\frac{4}{3}\pi a^3} = \frac{3}{5}ka^2.$$

The mean density is thus shown to be three-fifths the density at the surface of the sphere.

[The mean density of the earth according to the best determinations is very nearly 5.527 times that of water, while the average density of rocks at or near the surface is only about two and a half times that of water; hence, the mean density of the earth is about twice the average density at the surface.] See *Corollary* at end of Art. 190.

Example 2. — Find the mass and mean density of a semi-circular plate of radius a, whose density varies as the distance from the bounding diameter. Here $\gamma = ky$,

$$\therefore \quad m = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} ky\,dx\,dy = \tfrac{2}{3}ka^3;$$

$$\therefore \quad \bar{\gamma} = \frac{\frac{2}{3}ka^3}{\frac{1}{2}\pi a^2} = \frac{4\,ka}{3\,\pi}.$$

Or
$$m = \int_0^{\pi}\int_0^{a} k\rho^2 \sin\theta\,d\theta\,d\rho = \tfrac{2}{3}ka^3,$$

where $ky = kr\sin\theta$.

By a single integration, the element of area being

$$x \cdot \Delta y = \sqrt{a^2 - y^2}\,\Delta y,$$

$$m = 2\int_0^{a} k\sqrt{a^2 - y^2}\,y\,dy = \tfrac{2}{3}ka^3.$$

Example 3. — Find the mean density of a straight wire of length l, the density of which varies as the distance from one end.

$$\bar{\gamma} = \frac{\int_0^{l} ks\,ds}{l} = \frac{kl}{2}.$$

Example 4. — Find the mass and mean density of a hemi-spherical solid, radius a, the density varying as the distance from the base.

$$m = \int_0^{a} kz\pi x^2\,dz = \int_0^{a} k\pi\,(a^2 - z^2)\,z\,dz = \tfrac{1}{4}\pi ka^4;$$

$$\therefore \quad \gamma = \frac{\frac{1}{4}\pi ka^4}{\frac{2}{3}\pi a^3} = \frac{3}{8}ka.$$

Here the element of volume is a spherical segment, $\pi x^2\,\Delta z = \pi\,(a^2 - z^2)\,\Delta z$, at a distance z from the base.

Example 5. — Find the mean density of a right circular

cone of height h, in which the density varies as the distance from a plane through the vertex perpendicular to the axis.

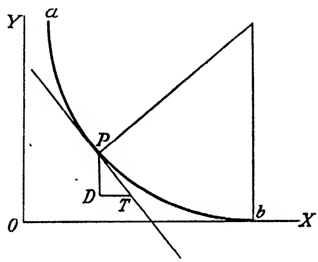
$$\bar{\gamma} = \frac{\int kz\pi x^2\, dz}{\frac{1}{3}\pi a^2 h} = \frac{k\pi \frac{a^2}{h^2}\int_0^h z^3\, dz}{\frac{1}{3}\pi a^2 h} = \frac{3}{4}kh.$$

Here origin is taken at the vertex and element of volume is

$$\pi x^2 \Delta z = \pi \frac{a^2}{h^2} z^2,$$

a being radius of base; $\gamma = kz$.

184. Curvilinear Motion. — Let a body slide without friction down any curve ab. The acceleration caused by gravity at any point P is $g \sin \alpha$, where $\alpha = PTD$, PT being a tangent to the curve.



Let $PT = ds$; then $-PD = dy$; hence,

$$\frac{d^2s}{dt^2} = g \sin \alpha = -g\frac{dy}{ds}. \qquad (1)$$

Let y_0 be the ordinate of the initial point on the curve; then $v = 0$ when $y = y_0$.

Integrating (1) gives

$$v = \frac{ds}{dt} = \sqrt{2g(y_0 - y)}. \qquad (2)$$

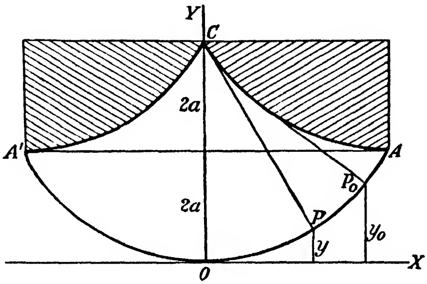
It follows from (2) that the velocity of a body acquired by moving freely down *any* frictionless path is the same, and is what it would acquire in falling freely through the vertical height between the initial and terminal points. Since $t = \int \frac{ds}{v}$, the time will depend upon the path.

* **Simple Circular Pendulum.** — Consider the motion of a particle on a smooth circular arc under the action of gravity as the only force. The value $\pi\sqrt{\frac{a}{g}}$ is the expres-

* See Art. 236, Applied Calculus.

sion usually taken as the time of a vibration; $2\pi\sqrt{\dfrac{a}{g}}$ being the value taken for a complete oscillation back and forth.

185. Cycloidal Pendulum. — A particle moves along the arc of a cycloid; find the time of descent.



From (2) of Art. 184,

$$v = \frac{ds}{dt} = \sqrt{2g(y_0 - y)}. \tag{1}$$

The equation of the cycloid referred to OX and OY is

$$x = a \operatorname{vers}^{-1} y/a + \sqrt{2ay - y^2}.$$

Hence, $ds = -\sqrt{2a/y}\,dy$, which substituted in (1), gives

$$t = \sqrt{\frac{a}{g}} \int_y^{y_0} \frac{dy}{\sqrt{y_0 y - y^2}} = \sqrt{\frac{a}{g}} \operatorname{vers}^{-1} \frac{2y}{y_0}\Big]_y^{y_0} = \sqrt{\frac{a}{g}} \left[\pi - \operatorname{vers}^{-1} \frac{2y}{y_0}\right];$$

$$\therefore \quad t = \pi\sqrt{\frac{a}{g}}, \quad \text{when} \quad y = 0, \quad \text{and} \quad 2t = T = 2\pi\sqrt{\frac{a}{g}},$$

the time of one oscillation of a pendulum if it swings in the arc of a cycloid.

The time of an oscillation being independent of the length of the arc, the cycloidal pendulum is isochronal.

The pendulum is described in Art. 97, Ex. 3.

The cycloid is the curve of quickest descent, *the Brachystochrone;* that is, the curve down which without friction gravity will cause a particle to fall in the shortest time.

APPLICATIONS.

186. Rectilinear Motion. — I. When the acceleration is constant,

$$\frac{da}{dt} = \frac{d^3s}{dt^3} = 0; \quad \frac{d^2s}{dt^2} = a; \quad v = \frac{ds}{dt} = at + v_0;$$

$$s = \tfrac{1}{2}at^2 + v_0t + s_0.$$

For bodies falling freely towards the earth from moderate heights, the acceleration g being taken constant,

$$v = v_0 - gt, \quad s = v_0t - \tfrac{1}{2}gt^2 + s_0;$$

from rest,

$$v = -gt, \quad s = -\tfrac{1}{2}gt^2. \quad \text{(Ex. 5, Art. 115.)}$$

Projected outward from rest, $h = v_0t - \frac{1}{2}gt^2$ (Ex. 6, Art. 115), where h is height from point of projection.

II. When the acceleration varies as the distance. Let $a = \dfrac{d^2s}{dt^2} = -ks$, where k, a constant, is the acceleration at a unit's distance from the origin; and let the body be of unit mass at an initial distance r; then,

$$2\left(\frac{ds}{dt}\right)d\left(\frac{ds}{dt}\right) = -2\,ks\,ds;$$

$$v^2 = \left(\frac{ds}{dt}\right)^2 = C_1 - ks^2 = kr^2 - ks^2, \text{ where } kr^2 = C_1;$$

$$v = \frac{ds}{dt} = k^{\frac{1}{2}}\sqrt{r^2 - s^2};$$

$$t = k^{-\frac{1}{2}}\int\frac{ds}{\sqrt{r^2 - s^2}} = k^{-\frac{1}{2}}\cos^{-1}\frac{s}{r}\,(+\,C_2 = 0),$$

$$t = 0, \text{ when } s = r;$$

whence

$$s = r\cos\left(k^{\frac{1}{2}}t\right) = r\sin\left(k^{\frac{1}{2}}t + \frac{\pi}{2}\right),$$

where the last result is gotten if the positive sign of the radical is taken in integrating.

Putting $s = 0$, gives $v = r\sqrt{k}$, the velocity at the origin, and $t = \pi/2\,k^{-\frac{1}{2}}$, $\frac{3}{2}\,\pi k^{-\frac{1}{2}}$, $\frac{5}{2}\,\pi k^{-\frac{1}{2}}$, . . . , or $s = r$, gives $t = 0$, $2\,\pi/k^{\frac{1}{2}}$, $4\,\pi/k^{\frac{1}{2}}$, . . . Hence the motion is periodic, the period being $2\,\pi/k^{\frac{1}{2}}$, which is independent of the initial distance.

Differentiating $s = r \cos (k^{\frac{1}{2}}t)$, gives

$$v = \frac{ds}{dt} = -k^{\frac{1}{2}}r \sin (k^{\frac{1}{2}}t),$$

which expresses the velocity in terms of the time that the body has been moving.

It is seen that this is a case of *simple harmonic motion.* (Art. 73 and Art. 226, Applied Calculus.)

It has been shown in Art. 190, *Cor.*, that the attraction of a sphere of uniform density for an internal particle varies as the distance from the center. Hence, a particular case of the periodic motion just considered would be that of a body which could pass freely through the earth, taken as a homogeneous sphere. Such a body would vibrate through the center from surface to surface. To find the time of this half period:

$$\begin{aligned} t = \pi k^{-\frac{1}{2}} &= 3.1416\sqrt{20900000/32.17}\text{ sec.} \\ &= 42\text{ min. about.} \end{aligned}$$

III. When the acceleration varies inversely as the square of the distance.

Let k be the acceleration at unit distance from origin;

then, $$a = \frac{d^2s}{dt^2} = -\frac{k}{s^2},$$

where s is to be taken always positive; multiplying by $2\,ds$,

$$2\left(\frac{ds}{dt}\right) d\left(\frac{ds}{dt}\right) = -2\,ks^{-2}\,ds;$$

integrating, $$v^2 = \left(\frac{ds}{dt}\right)^2 = 2\,k\left(\frac{1}{s} - \frac{1}{r}\right), \tag{1}$$

where $s = r$, when $v = 0$, which gives the velocity of a particle at any distance s.

For the time,

$$\sqrt{\frac{2k}{r}}\,dt = \frac{-s\,ds}{\sqrt{rs - s^2}},$$

negative, since s decreases as t increases,

$$= \left[\frac{1}{2}\frac{r - 2s}{\sqrt{rs - s^2}} - \frac{r}{2}\frac{1}{\sqrt{rs - s^2}}\right] ds;$$

integrating between limits corresponding to $t = t$ and $t = 0$, gives

$$t = \sqrt{\frac{r}{2k}}\left[\sqrt{rs - s^2} - \frac{r}{2}\text{vers}^{-1}\frac{2s}{r} + \frac{\pi r}{2}\right]. \qquad (2)$$

When the particle arrives at the origin, $s = 0$, therefore, the time to the origin from the point where $s = r$ is

$$t = \frac{\pi}{\sqrt{k}}\left(\frac{r}{2}\right)^{\frac{3}{2}}.$$

It is seen from (1) that the velocity $= 0$ when $s = r$, and $= \infty$ when $s = 0$; hence the particle approaches the origin with increasing velocity. While the attractive force causing the acceleration is very great near the origin, there can be no attraction *at* the origin itself; therefore, the particle must pass through the origin; and the conditions being the same on either side of the origin the motion must be retarded as rapidly as it was accelerated; hence, the particle will go to a point at a distance r equal to that from which it started and the motion will continue oscillatory.

An illustration of this general case has been given in Art. 193, where the attraction of the Earth for an external particle was considered as the cause of motion.

IV.* When the acceleration varies as the distance and the motion is away from the origin.

* See Art. 234, Applied Calculus.

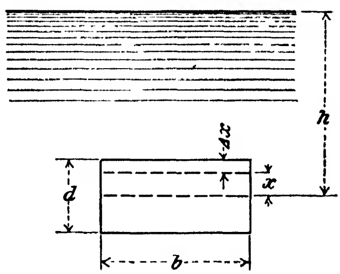
187. Discharge from an Orifice. — The coefficient of velocity $c_v = \dfrac{\text{actual velocity}}{\text{theoretical velocity}}$; the coefficient of velocity for a small sharp edge orifice is 0.98; and, for a short tube, it is about 0.82. If a is the area of a small orifice, then the theoretical discharge, or the quantity of liquid issuing in a unit of time, is

$$Q = av = a\sqrt{2gh}. \tag{1}$$

On account of the contraction of the jet at the orifice and the diminution of the velocity the actual discharge is

$$Q = c_c c_v av = 0.6\, a\sqrt{2gh}, \tag{2}$$

where $c_d = c_c c_v = 0.62 \times 0.98 = 0.6$ about, for a standard orifice; for a standard short tube, $c_d = 0.82$, the coefficient of contraction being unity.



For a small orifice the head is taken as constant and as that on the center, and for heads greater than twice the height of the orifice that gives the discharge almost exactly. For large orifices under low heads the variation of head over the orifice, causing a variation in the velocity of the jet and therefore in the discharge, makes the formulas above inapplicable for exact results.

Let h be the head on the center of a rectangular orifice of breadth b and depth d; and let the rectangle be supposed

to be divided into horizontal strips of area $b\,\Delta x$, x being the distance from the center line of the rectangle. The quantity

$$Q = \lim_{\Delta x \doteq 0} \sum b\,\Delta x\,\sqrt{2g(h-x)} = b\sqrt{2g}\int_{-\frac{d}{2}}^{\frac{d}{2}} (h-x)^{\frac{1}{2}}\,dx;$$

$$Q = b\sqrt{2gh}\int_{-\frac{d}{2}}^{\frac{d}{2}} \left(1 - \frac{x}{2h} - \frac{x^2}{8h^2} - \cdots\right)dx \quad \left(\begin{array}{l}\text{by expand-}\\ \text{ing in series}\end{array}\right)$$

$$= bd\sqrt{2gh}\left(1 - \frac{d^2}{96\,h^2} - \frac{1}{2048}\frac{d^4}{h^4} - \cdots\right). \tag{3}$$

It is seen that the quantity in the parenthesis is less than unity, and the discharge is therefore less than that given by (1).

For $h = 2\,d$, the value of the parenthesis factor is 0.997, so for heads greater than twice the height of the rectangle the discharge may be figured from $Q = ca\sqrt{2\,gh}$, where c is the coefficient of discharge.

Integrating without expanding $(h-x)^{\frac{1}{2}}$ gives

$$Q = b\sqrt{2g}\int (h-x)^{\frac{1}{2}}\,dx = b\sqrt{2g}\left[-\tfrac{2}{3}(h-x)^{\frac{3}{2}}\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= \frac{2}{3}b\sqrt{2g}\left[\left(h+\frac{d}{2}\right)^{\frac{3}{2}} - \left(h-\frac{d}{2}\right)^{\frac{3}{2}}\right]. \tag{4}$$

If the orifice extends to the surface and the bottom is h below,

$$Q = b\sqrt{2g}\left[-\tfrac{2}{3}(h-x)^{\frac{3}{2}}\right]_0^h = \tfrac{2}{3}\,bh\sqrt{2\,gh}, \tag{5}$$

which is just $\frac{2}{3}$ the quantity that would flow through an orifice of equal area placed horizontally at the depth h, the vessel being kept constantly full.

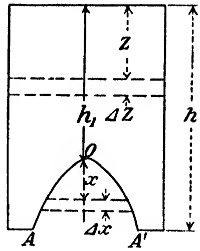
The mean velocity v_m is seen in (5) to be $\frac{2}{3}\sqrt{2\,gh}$.

For any vertical orifice formed by a plane curve whose vertex 0 is at the depth h_1 below the surface of the liquid in

a vessel of height h, kept constantly full, the formula for discharge is

$$Q = \int_0^{h-h_1} 2y\sqrt{2g(h_1+x)}\,dx. \tag{6}$$

To get the time of emptying the vessel; let the surface be z below the top at the end of the time t, $z = 0$ when $t = 0$; then the quantity discharged in an element of time is



$$dQ = \left[2\sqrt{2g}\int_0^{h-h_1} y\sqrt{x+h_1-z}\,dx\right]dt,$$

z being constant during this integration; and since in the same time the quantity discharged through the orifice must be $A\,dz$, A being the area of the section of the vessel at depth z, it follows that

$$\left[2\sqrt{2g}\int y\sqrt{x+h_1-z}\,dx\right]dt = A\,dz;$$

$$\therefore \quad t = \frac{1}{2\sqrt{2g}}\int_0^h \frac{A\,dz}{\int_0^{h-h_1} y\sqrt{x+h_1-z}\,dx}. \tag{7}$$

Example. — Water is flowing from an orifice in the side of a cylindrical tank whose cross section is 100 sq. ft. The velocity of the jet is $\sqrt{2gx}$, x being the height of the surface above the orifice; and the cross section of the jet is 0.01 sq. ft. Find the time it will take for the water to fall from 100 ft. to 81 ft. above the orifice.

For this example the formula becomes

$$t = -\frac{A}{ca\sqrt{2g}}\int_h^{h_1} x^{-\frac{1}{2}}\,dx \text{ (where } x \text{ is height of surface}$$

above orifice and a is area of the orifice)

$$= -\frac{100}{0.01\sqrt{2g}}\int_{100}^{81} x^{-\frac{1}{2}}\,dx = -\frac{10000}{8}\cdot 2h^{\frac{1}{2}}\Big]_{100}^{81} \quad \begin{pmatrix}\text{taking}\\ g = 32\end{pmatrix}$$

$$= 2500(10-9) = 2500 \text{ sec.} = 41\tfrac{2}{3} \text{ min.}$$

CHAPTER VI.

APPLICATIONS. PRESSURE. STRESS. ATTRACTION.

188. Intensity of a Distributed Force. — A distributed force is one that acts on a surface, such as the pressure of water against the surface of contact, the pressure of a weight upon the surface of its support; or, one that acts through a given volume, such as the attraction of the earth on a body.

All forces are really distributed forces since no finite force can act at a point of no area; although this is true, in some cases it is convenient to regard a force, whose place of application is small, as though it were applied at a point. Such a force is called a *concentrated force.* A distributed force is conceived as "equivalent to" a concentrated force called the resultant force, when the force of gravity acting on every particle of a body is taken as acting at a point within the body, called the center of gravity. A distributed force is regarded as the limiting case of a system of concentrated forces whose number becomes larger as their individual magnitudes become smaller. It is thus that a force is regarded as having a definite *point of application* and a definite *line of action:* when so regarded it is a *localized vector quantity.* When a force is distributed over an area, the *intensity* of the force at a point is the number of units of force acting on a unit of area including that point.

Briefly the intensity is defined as the force per unit of area. If the force is uniformly distributed, the intensity p will be equal to the force P, acting on the entire area, divided by the area A; that is,

$$p = \frac{P}{A}. \tag{1}$$

If the force is not uniformly distributed, the intensity at any point of the area will be given by

$$p = \lim_{\Delta A \doteq 0} \left[\frac{\Delta P}{\Delta A}\right] = \frac{dP}{dA}, \tag{2}$$

the limit of the ratio of the force, acting on a small element of the area, to that element as it approaches zero as a limit. When the intensity varies from point to point over any area, the force on that area divided by that area gives the *average* intensity on the area. In any case the entire force is given by

$$P = \int p\,dA, \tag{3}$$

where p, if variable, must be expressed in the same terms as dA in order to get P by integration.

If p is constant,

$$P = p\int dA = pA, \quad \text{and} \quad p = \frac{P}{A}. \tag{1}$$

189. Pressure of Liquids. — The pressure of a liquid on a surface is normal to the surface, and the intensity of pressure varies as the depth of the point below the free surface of the liquid. The intensity is given by

$$p = wh, \tag{1}$$

where w is the weight of a cubic unit of the liquid and h, called the *head*, is the depth of the point below the free surface.

If w is expressed in pounds per cubic foot, h should be in feet, and p will then result in pounds per square foot. For water w is usually taken as $62\frac{1}{2}$ lbs. per cubic foot, and the intensity of pressure given in pounds per square foot. When the intensity p is constant on any horizontally immersed plane surface the total pressure P is, by (1) Art. 188,

$$P = Awh = 62.5\,Ah \text{ lbs.} \tag{2}$$

When the surface under pressure is not horizontal, by (3), Art. 188,

$$P = \int_{h_1}^{h_2} wx\,dA, \tag{3}$$

where the limits of x are the least and greatest heads on the area. When the area extends to the surface of the liquid, the lower limit becomes zero and the upper may be taken as h.

Since in (3), $\int x\,dA = \bar{x}A,$ by (2), Art. 175, (Applied Calculus.)

$$P = \int wx\,dA = Aw\bar{x} = 62.5\,A\bar{x}\text{ lbs.}$$

Hence, *the total pressure on an immersed area is the product of that area, the weight of a cubic unit of water, and the head upon its center of gravity.*

In general, *the pressure of any liquid upon an area is equal to the weight of a column of liquid whose base is the area pressed and whose height is the depth of the center of gravity of the area below the surface.*

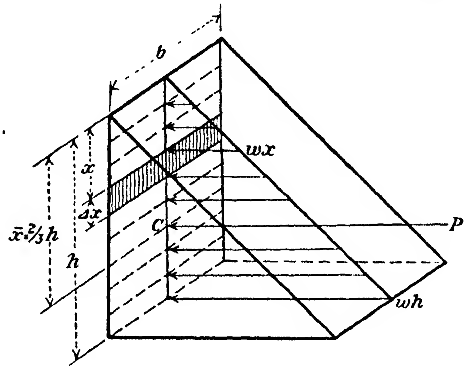
Example 1. — The vertical face of a dam subjected to the pressure of water is h ft. in height and b ft. in breadth. The pressure of the water varies as the depth; the intensity at a depth x is wx, w being the constant weight of a cubic unit of water. Required the total pressure on the face of the dam, and the location of *the center of pressure.*

Let the area of pressure be divided into strips of width Δx and length b, then $wx \cdot b\,\Delta x$ is *approximately* the pressure on the element of area — for wx is the intensity of pressure at the top of the strip.

The sum of a finite number of terms of the form $wbx\,\Delta x$ would give a result for the total pressure less than the actual value; but the exact value is

$$P = \lim_{\Delta x \doteq 0} \sum_0^h wbx\,\Delta x = wb\int_0^h x\,dx = \frac{wbx^2}{2}\Big]_0^h = \frac{wbh^2}{2} = \frac{wh}{2}\cdot bh. \tag{1}$$

The intensity of pressure is a uniformly varying force having zero value at the surface of the water and value wh at the bottom. The center of pressure, being the point of application of the resultant pressure, is given by taking the moment



of P about the surface line equal to the limit of the sum of the moments of the elementary pressures about that line:

$$\bar{x} \cdot P = \int_0^h wbx^2\,dx = \frac{wbx^3}{3}\bigg]_0^h = \frac{wbh^3}{3};$$

$$\therefore \quad \bar{x} = \frac{\int_0^h wbx^2\,dx}{\int_0^h wbx\,dx} = \frac{wbh^3/3}{wbh^2/2} = \frac{2}{3}h. \qquad (2)$$

In general, the center of pressure of a rectangle with a side at the surface is two-thirds the height of the rectangle below the surface. When the top of the area is h_1 below the surface and the bottom is h_2 below, the total pressure is

$$P = wb\int_{h_1}^{h_2} x\,dx = \frac{wb}{2}(h_2^2 - h_1^2),$$

and

$$\bar{x} = \frac{wb\int_{h_1}^{h_2} x^2\,dx}{wb\int_{h_1}^{h_2} x\,dx} = \frac{I}{M} = \frac{\frac{x^3}{3}\Big]_{h_1}^{h_2}}{\frac{x^2}{2}\Big]_{h_1}^{h_2}} = \frac{2}{3}\,\frac{h_2^3 - h_1^3}{h_2^2 - h_1^2}. \qquad (3)$$

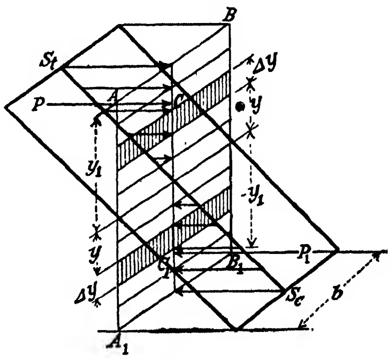
It may be noted that the second moment in the numerator is the *moment of inertia* of the area, and the first moment in the denominator is the *statical moment.*

Note. — That $P = \int wbx\,dx$ in (1) is the reversal of a rate may be seen by considering the rate of change of the total pressure when the depth x is increased by Δx, for then the pressure P on the area is increased by $\Delta P = wx \cdot b\,\Delta x$, approximately, and $\Delta P/\Delta x = wbx$ (nearly).

Hence $$\lim_{\Delta x \doteq 0} \frac{\Delta P}{\Delta x} = \frac{dP}{dx} = wbx,$$

the rate of change of P; and $P = \int_0^h wbx\,dx$ as in (1); so the total pressure is a function of h and its rate of change is $wbx = pb$, where $p = wx$ is the pressure on a unit of area.

Note. — Whenever an external force acts on a body it induces a resisting force within the body. This is in accordance with Newton's third law of motion. This internal resistance is due to the molecular forces or *stresses* within the body. A *stress* is a distributed force acting on a surface



Example 2. — A vertical rectangular section ABB_1A_1 of a beam of breadth b and depth d is subjected to a stress of tension and compression uniformly varying in intensity from zero at the middle fiber to S_t and S_c at the outside fibers, at the distance, $y_1 = \frac{1}{2}d$, from the neutral axis or middle fiber of the section.

Find the total tensile and compressive stresses and the centers of stress on each half of the section.

The intensity of stress at the distance y from the neutral axis is $(S/y_1)y$, hence $(S/y_1)yb\,\Delta y$ is approximately the stress on a strip Δy in depth — the intensity at the edge of the strip being taken. The sum of a finite number of such terms would give a result less than the total stress on the half section; but the exact value is given by

$$P = \lim_{\Delta y \doteq 0} \sum_0^{y_1} \frac{S}{y_1} yb\,\Delta y = \frac{Sb}{y_1}\int_0^{y_1} y\,dy$$

$$= \frac{Sb}{y_1}\left.\frac{y^2}{2}\right]_0^{y_1} = \left.\frac{Sb}{2}y_1\right]^{\frac{d}{2}} = \frac{Sbd}{4}.$$

For the center of stress, the point of application of the total stress,

$$\bar{y}\cdot P = \int_0^{y_1}\frac{Sb}{y_1}y^2\,dy = \frac{Sb}{y_1}\left.\frac{y^3}{3}\right]_0^{y_1} = \left.\frac{Sb}{3}y_1^2\right]^{\frac{d}{2}} = \frac{Sbd^2}{12};$$

$$\therefore \quad \bar{y} = \frac{Sbd^2}{12}\Big/\frac{Sbd}{4} = \frac{1}{3}d.$$

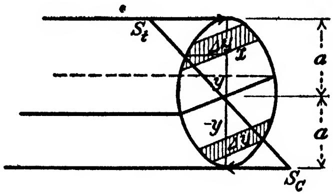
When $S = S_t = S_c$, $P = P_1$; hence, the total tensile and compressive stresses form a couple with arm $\frac{2}{3}\,d$, the moment being

$$\frac{Sbd}{4}\cdot\frac{2}{3}d = \frac{Sbd^2}{6}, \text{ called the } \textit{section modulus.}$$

By the mechanics of beams, if M denote the moment of the external forces acting on the beam, $M = \dfrac{Sbd^2}{6}$.

Example 3. — A vertical circular section of a beam is subjected to a stress of tension and compression uniformly varying in intensity from zero at the horizontal diameter $2\,a$ of the section to S_t and S_c at the top and bottom fibers, respectively.

Find the total stresses on the upper and lower semicircles and the centers of stress on the semicircles.



Denoting by P the total stress on either semicircle, and taking $S_t = S_c = S$;

$$P = \lim_{\Delta y \doteq 0} \sum_0^a \frac{S}{a} y \cdot 2x\,\Delta y = \frac{2S}{a}\int_0^a \sqrt{a^2 - y^2}\, y\, dy$$

$$= \frac{2S}{a}\left[-\frac{1}{3}(a^2 - y^2)^{\frac{3}{2}}\right]_0^a = \frac{2S}{3a} \cdot a^3 = \frac{2}{3} Sa^2 = \frac{1}{6} S\, d^2;$$

$$\bar{y} \cdot P = \frac{2S}{a}\int_0^a \sqrt{a^2 - y^2}\, y^2 dy$$

$$= \frac{2S}{a}\left[\frac{y}{8}(2y^2 - a^2)\sqrt{a^2 - y^2} + \frac{a^4}{8}\sin^{-1}\frac{y}{a}\right]_0^a \left(\begin{matrix}\text{See Ex. 2, Exer-}\\ \text{cise XXV.}\end{matrix}\right)$$

$$= \frac{2S}{a} \cdot \frac{\pi a^4}{16} = \frac{S\pi a^3}{8} = \frac{S\pi d^3}{64};$$

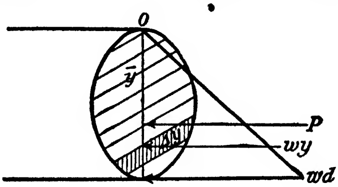
$$\therefore \quad \bar{y} = \frac{S\pi d^3}{64} \Big/ \frac{Sd^2}{6} = \frac{3\pi d}{32}.$$

Hence the couple formed by the forces P has an arm, $2\bar{y} = \frac{3}{16}\pi d$, and

$$M = \frac{1}{6} Sd^2 \cdot \frac{3}{16}\pi d = \frac{S\pi d^3}{32}, \text{ the } \textit{section modulus.}$$

Example 4. — Find the total water pressure upon the end of a circular right cylinder immersed lengthwise, one element of the cylinder just at the surface of the water. Find the center of pressure of the circular area.

The intensity of pressure at a depth y being wy, the approximate pressure on a strip is $wy \cdot 2x\,\Delta y$.



The total pressure is $P = \lim\limits_{\Delta y \doteq 0} \sum_0^{2a} 2wxy\,\Delta y$;

$$\therefore \quad P = 2w\int_0^{2a} (2ay - y^2)^{\frac{1}{2}} y\,dy = 2wa\int_0^{2a} (2ay - y^2)^{\frac{1}{2}}\,dy$$

$$= 2wa \cdot \frac{\pi a^2}{2} = w\pi a^3; \quad \text{(Ex. 13, Exercise XXV.)}$$

$$\therefore \quad \bar{y} = \frac{2w\int_0^{2a}(2ay - y^2)\,y^2 dy}{P} = \frac{\frac{5}{4}a \cdot 2w\int_0^{2a}(2ay-y^2)^{\frac{1}{2}}\,y\,dy}{w\pi a^3}$$

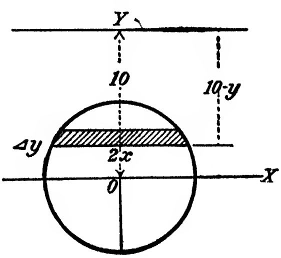
$$= \frac{\frac{5}{4}w\pi a^4}{w\pi a^3} = \frac{5}{4}a = \frac{5}{8}d. \quad \text{(Ex. 13, Exercise XXV.)}$$

EXERCISE XLIII.

1. (*b*) The pressure upon one side of the gate of a dry dock, the wetted area being a rectangle 80 ft. long and 30 ft. deep, is to be found exactly. Take $w = 62\frac{1}{2}$ lb. for the weight of a cubic foot of water.

(*c*) Find the depth of the center of pressure. *Ans.* (*c*) 20 ft.

(*a*) Find the pressure approximately by a limited number of terms. (See Art. 154.) *Ans.* (*b*) $112\frac{1}{2}$ tons.



2. The pressure on the gate that closes a water main half full of water, the diameter of the main being 8 ft. Get the exact (*b*) pressure only. (*c*) Find the center of pressure.

Ans. (*b*) $P = \frac{128}{3}\,w$ lbs. (*c*) $\bar{y} = \frac{3}{4}\pi$ ft.

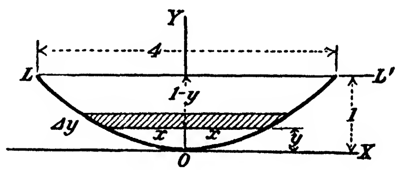
3. Find the exact pressure on a circular disk 10 ft. in diameter, submerged below water with its plane vertical and its center 10 ft. below the surface. Here

$$P = w\int_{-5=-a}^{5=a}(10-y)\,2x\,dy = 2w\int_{-5=-a}^{5=a}(10-y)(a^2-y^2)^{\frac{1}{2}}\,dy$$
$$= 20\,w\int_{-5}^{5}(a^2-y^2)^{\frac{1}{2}}\,dy - 2\,w\int_{-5}^{5}(a^2-y^2)^{\frac{1}{2}}\,y\,dy$$
$$= 20\,w\cdot\tfrac{25}{2}\pi + \tfrac{2}{3}(a^2-y^2)^{\frac{3}{2}}\Big]_{-5=-a}^{5=a} = 250\,\pi w$$
$$= 250\,\pi\cdot 62\tfrac{1}{2}\text{ lb.}$$

4. Find the pressure on the face of a temporary bulkhead 4 ft. in diameter closing an unfinished water main, when water is let in from the reservoir. The center of bulkhead is 40 ft. below the surface of the water in the reservoir. *Ans.* Nearly 16 tons.

5. Find the pressure on the end of a parabolic trough when it is full of water. The parabola has its vertex downward, the latus rectum is in the surface and is 4 ft. long. Here

$$P = \lim_{\Delta y \doteq 0}\sum_0^1 2\,w(1-y)\,x\,\Delta y = 2\,w\int_0^1 2(1-y)\,y^{\frac{1}{2}}\,dy$$
$$= 4\,w\int_0^1 y^{\frac{1}{2}}\,dy - \int_0^1 y^{\frac{3}{2}}\,dy$$
$$= 4\,w\left[\tfrac{2}{3}y^{\frac{3}{2}} - \tfrac{2}{5}y^{\frac{5}{2}}\right]_0^1 = \tfrac{16}{15}\,w = 66\tfrac{2}{3}\text{ lbs.}$$



6. A horizontal cylindrical tank is half full of oil weighing 50 lb. per cubic foot. The diameter of each end is 4 ft. Find the pressure on each end. Find the pressure when the tank is full also.

Ans. $266\frac{2}{3}$ lb.; 1256 lb.

190. *Attraction. Law of Gravitation. — Every portion of matter acts on every other portion of matter with forces of attraction or repulsion. According to Newton's Law of Universal Gravitation, every particle of matter attracts every other such particle with a force which acts along the line joining the two particles, and whose magnitude is pro-

* This article is based on a discussion in Fuller and Johnston's *Applied Mechanics.*

portional directly to the product of their masses and inversely to the square of the distance between them.

If the masses of the particles are m and m_1 and the distance between them is r, the law may be expressed algebraically by

$$F = K\frac{mm_1}{r^2}, \tag{1}$$

where F is the attractive force between the particles and K is a constant, determined by experiment, its numerical value depending on the units in which F, m, m_1 and r are expressed. The value of K having been determined in one case is then known for all cases.

While formula (1) expresses the law of gravitation, the general algebraic expression for the law of attraction would be

$$F = K\frac{mm_1}{\phi(r)}, \tag{2}$$

where $\phi(r)$ is some function of the distance between the particles, depending on the nature of the attractive force, K is a constant, and m and m_1 other quantities than the masses of particles.

In interpreting formula (1), it is to be noted that it applies strictly only to particles; for the particles having finite masses must have finite dimensions and hence, as the distance between them is diminished, r cannot be less than a certain finite quantity and the maximum value of F, when the particles are in contact, will be a finite quantity. If r were taken to be zero in any case, F for finite values of m and m_1 would become ∞ which would be impossible under the conditions.

The formula, while applying strictly only to particles, gives, to a close approximation, the attraction between two bodies of finite size, whose linear dimensions are small compared to the distance between them. In the application of the law the attraction of one particle on another may be

regarded as acting at a point. It will be shown that any sphere attracts any outside particle as if the whole attraction was towards a point at the center of the sphere, but, in general, the attraction of bodies on exterior particles is not always towards the center of gravity of the attracting body.

Attraction of gravitation is a *mutual* action between two particles or bodies; that is, each exerts an attractive force upon the other, the two forces being equal in magnitude and opposite in direction. This is implied in the Law, and it is also in accordance with the law of "action and reaction," Newton's third law of motion.

It is evident that, in formula (1), K is equal to the force with which two particles of unit mass at a unit distance apart attract each other.

If the equation is divided by m_1, then

$$\frac{F}{m_1} = a = K\frac{m}{r^2}, \tag{3}$$

where a is the acceleration which would be produced in the mass m_1 by the attraction of the mass m at a distance r.

The quantity $K\frac{m}{r^2}$ would also equal the force of attraction exerted by the mass m on a mass *unity* at a distance r. Briefly this is called the *attraction at the point*, at which the unit mass is situated, exerted by the mass m.

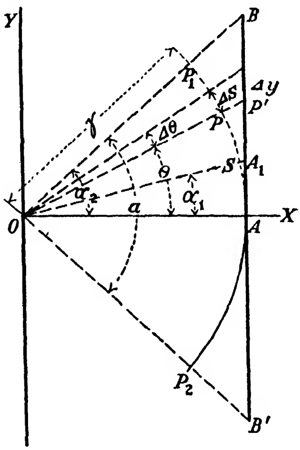
The attraction at a point exerted by any mass is called the *strength of the field of force*, or briefly, the *strength of field*, by which the space through which the attraction of the mass is exerted is expressed.

Electrostatic and magnetic attraction and repulsion are other examples of forces, which are governed by laws similar to that of gravitation.

The following examples are based on the law

$$F = K\frac{m}{r^2},$$

where F is the attraction of a particle of mass m for a particle of unit mass, the body being taken as *homogeneous*, of uniform density; that is, each cubic unit having the same weight.



Example 1. — *Attraction of a Rod of Uniform Section.* — (*a*) Let the rod of small section be in the form of a circular arc; to find the attraction at the center of the circle.

Let r be the radius, α the angle subtended at the center, and m the mass of a unit length of the rod. Take the axis OX bisecting the angle α, and let θ be the angle which the radius to any point P makes with OX. The attraction at O of a particle at P is

$$\Delta F = \frac{Km\,\Delta s}{r^2} = \frac{Km\,\Delta\theta}{r}. \qquad (1)$$

Since all the elementary forces of attraction are directed to the point O, the resultant R is found from the sum of the components of the elementary forces.

$$\sum X = \frac{Km}{r}\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \cos\theta\, d\theta = \frac{2\,Km}{r}\sin\frac{\alpha}{2},$$

$$\sum Y = \frac{Km}{r}\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \sin\theta\, d\theta = 0,$$

the attractions being neutralized.

Hence, $$R = \sqrt{(\sum X)^2 + (\sum Y)^2} = \frac{2\,Km}{r}\sin\frac{\alpha}{2}. \qquad (2)$$

When $\alpha = \pi$, $$R = \frac{2\,Km}{r}. \qquad (3)$$

When $\alpha = 2\pi$, $R = 0$; since the arc being a circumference of a circle, the attractions neutralize each other.

(*b*) Let the rod be straight; to find the attraction at a point. Let r be the shortest distance from a point O to the rod. Taking O as origin, the equation of the rod is $x = r$ (constant).

When the rod is $B'AB$ the angles may be taken as in (*a*) for the circular arc. The attraction at O of a particle at P' on the rod is

$$\Delta F = \frac{Km}{(OP')^2}\Delta y = \frac{Km\cos^2\theta}{r^2}\Delta y. \tag{1'}$$

The resultant attraction is found as in (*a*); since $y = r\tan\theta$,

$$dy = \frac{r\,d\theta}{\cos^2\theta};$$

$$\sum X = \frac{Km}{r^2}\int\cos^3\theta\,dy = \frac{Km}{r}\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}}\cos\theta\,d\theta$$

$$= \frac{2\,Km}{r}\sin\frac{\alpha}{2}, \text{ as in (a).}$$

$$\sum Y = \frac{Km}{r^2}\int\cos^2\theta\sin\theta\,dy = \frac{Km}{r}\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}}\sin\theta\,d\theta = 0.$$

Hence, $$R = \frac{2\,Km}{r}\sin\frac{\alpha}{2}, \text{ as in (a).} \tag{2'}$$

It is thus shown that if the straight rod $B'AB$ is of the same mass per unit length as P_2P_1, the resultant attraction of $B'AB$ at O is the same as the attraction of P_2P_1, since the sum of the attractions of the elementary masses $m\,\Delta y$ and the sum of the attractions of the elementary masses $m\,\Delta s$ have the same limit.

(*b'*) Let the rod be still straight but A_1B, the angles with OX of the lines from O to the ends being α_1 and α_2; then,

$$\sum X = \frac{Km}{r}\int_{\alpha_1}^{\alpha_2}\cos\theta\,d\theta = \frac{Km}{r}(\sin\alpha_2 - \sin\alpha_1),$$

$$\sum Y = \frac{Km}{r}\int_{\alpha_1}^{\alpha_2}\sin\theta\,d\theta = \frac{Km}{r}(\cos\alpha_1 - \cos\alpha_2).$$

Hence,

$$R = \frac{Km}{r}\sqrt{2\left[1 - \cos\left(\alpha_2 - \alpha_1\right)\right]} = \frac{2Km}{r}\sin\frac{\alpha_2 - \alpha_1}{2}, \quad (3')$$

and

$$\tan\theta_r = \frac{\sum Y}{\sum X} = -\frac{\cos\alpha_2 - \cos\alpha_1}{\sin\alpha_2 - \sin\alpha_1} = \tan\frac{\alpha_2 + \alpha_1}{2},$$

$$\therefore \quad \theta_r = \frac{\alpha_2 + \alpha_1}{2},$$

the line of action of R bisecting the angle A_1OB, subtended by A_1B at O.

(b'') Let the point O be at A, making $r = 0$ and (3′) indeterminate. Then,

$$\Delta F = \frac{Km\,dy}{y^2},$$

$$\therefore \quad R = Km\int_{y_1}^{y_2}\frac{dy}{y^2} = Km\left(\frac{1}{y_1} - \frac{1}{y_2}\right) = \frac{Km\left(y_2 - y_1\right)}{y_2y_1} = \frac{KM}{y_2y_1}, \quad (4)$$

where M = the entire mass of the rod.

If the point O is taken at the end of the rod, $y_1 = 0$ and equation (4) gives $R = \infty$. This is impossible; for, as stated in Art. 190, r cannot be zero for finite particles. If, however, $y_2 = \infty$,

$$R = \frac{Km}{y_1},$$

making R a finite quantity for any length from A_1.

If the point O were taken on the rod between A_1 and B, with lower limit, $-y_1$,

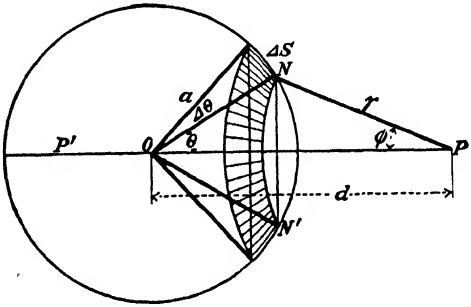
$$R = -Km\left(\frac{1}{y_1} + \frac{1}{y_2}\right),$$

and, if O were taken at the middle point of the rod, it is evident that $R = 0$.

Example 2. — *Attraction of a Spherical Shell at a Point.* — Find the resultant attraction of a spherical shell of uniform density and small uniform thickness on a particle of unit mass, M' being the mass of the shell.

(*a*) Let the point P outside the shell be the position of the particle.

Let γ be the density and t the thickness of the shell, O its center, and a the radius ON; let $NP = r$ and $OP = d$. If the circle be revolved about OP as an axis through an angle 2π, a thin spherical shell of thickness t will be generated, and an elementary volume will be generated by the elementary area at N, whose mass will be $\Delta M' = \gamma t \cdot 2\pi a^2 \sin\theta\,\Delta\theta$, approximately.



The attraction of the elementary mass at N for the particle of unit mass at P is

$$\Delta^2 F = \frac{K\gamma ta\,\Delta\theta}{r^2}. \qquad (1)$$

This attraction may be resolved at P into a component X along PO and a component Y perpendicular to PO. To every elementary mass at N there is a corresponding mass at N', whose attraction at P is X along PO, and $-Y$ perpendicular, which neutralizes Y. Hence the attraction of $\Delta M'$ is along PO, and is given approximately by

$$\Delta F = \frac{K\gamma t \cdot 2\pi a^2 \sin\theta\,\Delta\theta}{r^2}\cos\phi. \qquad (2)$$

From the geometry of the figure,

$$r^2 = a^2 + d^2 - 2\,ad\cos\theta,$$

which differentiated gives

$$r\,dr = ad \sin\theta\, d\theta; \quad \therefore \quad \sin\theta = \frac{r\,dr}{ad \cdot d\theta};$$

and from the figure, $\cos\phi = \dfrac{d - a\cos\theta}{r}$.

Substituting these values in (2) gives exactly,

$$dF = K\gamma\pi \frac{at}{d^2}\left(\frac{d^2 - a^2 + r^2}{r^2}\right) dr; \tag{3}$$

hence,

$$F = K\gamma\pi \frac{at}{d^2}\int_{d-a}^{d+a}\left(\frac{d^2 - a^2 + r^2}{r^2}\right) dr$$

$$= \frac{4\,K\gamma\pi a^2 t}{d^2} = \frac{KM'}{d^2}. \tag{4}$$

It follows from (4) that the attraction is the same as though the mass of the shell were concentrated at its center. It follows also that a sphere, which is either homogeneous or consists of concentric shells of uniform density, attracts a particle without the sphere as if the mass of the sphere were concentrated at its center. This law holds almost exactly, for bodies slightly flattened at the poles, if the particle is not too close to the attracting body. Since both these conditions exist in the case of the Earth and other members of the Solar System, this law has important applications.

(*b*) Let the point P' inside the shell be the position of the particle. The equation (3) in case (*a*) is true for this case too, but the limits for r are now $a - d$ and $a + d$. Hence,

$$F = \frac{K\gamma\pi at}{d^2}\left[\frac{a^2 - d^2}{r} + r\right]_{a-d}^{a+d} = 0; \tag{5}$$

that is, the resultant of all the attractions of the elementary masses of the spherical shell on a particle within the shell is zero.

(*c*) Let the point where the particle is be on the surface of the shell.

In (4) making $d = a$ gives

$$F = 4\,K\gamma\pi t = \frac{KM'}{a^2}. \tag{6}$$

Corollary. — If a particle be inside a homogeneous sphere at a distance d from its center, all that portion of the sphere at a greater distance from the center than the particle has no effect on the particle, while the remaining portion attracts the particle in the same way as if the mass of the remaining portion were concentrated at the center of the sphere. Thus the attraction of the sphere on the particle is

$$F = \frac{\frac{4}{3}K\pi\gamma d^3}{d^2} = \frac{4\,K\pi\gamma d}{3}; \tag{7}$$

that is, within a homogeneous sphere the attraction varies as the distance from the center. The attraction of a sphere of mass M on a particle at the surface is from (7), making $d = a$,

$$F = \frac{4}{3}K\pi\gamma a = \frac{KM}{a^2}. \tag{8}$$

Hence, the attraction for an external particle is

$$F = \frac{KM}{d^2}, \tag{9}$$

where d is the distance from the particle to center of sphere.

Note. — The propositions respecting the attraction of a uniform spherical shell on an external or internal particle were given by Newton (Principia, Lib. I, Prop. 70, 71).

It was in 1685, nineteen years after he had conceived the theory of universal gravitation, that he completed the* verification of the theory, by proving that a sphere in which the density depends only upon the distance from the center attracts an external particle as if the mass of the sphere were concentrated at its center. Thus was the great induction by this supplementary proposition finally established.

* See Art. 198.

Example 3. — *Attraction of the Earth.** — I. Find the relation between the attraction of the Earth on a body at the surface and at a point h feet above the surface.

Taking the Earth as a sphere whose density is a function of the distance from the center, R as the radius, and F and F' as the Earth's attraction upon the body at the surface and at h feet above the surface,

$$F/F' = (R + h)^2/R^2 \quad \text{(by Ex. 2, (8) and (9))},$$

or

$$F' = FR^2/(R + h)^2. \tag{1}$$

If h is a small fraction of R, then approximately,

$$F' = F(1 + h/R)^{-2} = F(1 - 2h/R). \tag{2}$$

Since the "weight" of a body is the force with which the Earth attracts it, the equations (1) and (2) give the relation between the weight of a body at the surface and at a height h feet above the surface. And, if g and g' are the values of the acceleration of gravity at the surface and at the point h feet above the surface, since $F/F' = g/g'$, the equations give the relation between g and g' also.

(*a*) Find approximately at what height above the surface will the weight of a body be $\frac{1}{10}$ of one per cent less than at the surface.

Taking the mean radius of the Earth as 20,902,000 ft.,

$$F'/F = 1 - 2h/R = 1 - 1/1000;$$

$$\therefore \quad h = \frac{R}{2000} = \frac{20{,}902{,}000}{2000} = 10{,}451 \text{ feet}.$$

Corollary. — A mass which at the surface weighs one pound at 10,451 ft. will weigh 0.999 lb.

(*b*) Find how much the value of g is changed by a change of elevation of one foot above the surface.

$$\frac{F'}{F} = \frac{g'}{g} = 1 - \frac{2h}{R} = 1 - \frac{2}{20{,}902{,}000} = 1 - 0.0000000957.$$

* This example is based on examples in Hoskins's *Theoretical Mechanics*.

The value of g for different latitudes and elevations is given by the following formula, in which g is in feet per second, l is the latitude, and h the elevation in feet above sea level:

$$g = 32.0894\,(1 + 0.005243 \sin^2 l)\,(1 - 0.0000000957\,h).$$

This gives

$g = 32.0894$ at the equator at sea level, and

$g = 32.174$ at 45° latitude at sea level;

this latter value, $g = 32.174$ ft. per sec. per sec. is the *standard* value.

II. Find the relation between the attraction of the Earth on a body at the surface and at a point h below the surface.

(*a*) Taking the Earth as a sphere of uniform density of radius R,

$$F''/F = (R - h)/R = 1 - h/R \quad \text{(by } \textit{Cor.} \text{ Ex. 2)}, \tag{3}$$

where F and F'' denote the attraction at the surface and at h below the surface.

Corollary. — Under these conditions, the weight of a body and the value of g would decrease with the depth h below the surface.

(*b*) Taking the Earth as a sphere whose density is a function of the distance from the center, let γ denote the mean density of the whole Earth and γ_0 the mean density of the outer shell of thickness h.

Let M be the mass of the whole Earth, M'' that of the inner sphere of radius $R - h$, m the mass of the attracted body, F and F'' the attraction at the surface and at h below the surface. Then F is equal to the attraction between two particles of masses M and m whose distance apart is R, and F'' is equal to the attraction between two particles of masses M'' and m whose distance apart is $R - h$. That is,

$$F = KMm/R^2, \quad F'' = KM''m/(R - h)^2;$$

hence,

$$\frac{F''}{F} = \frac{M''}{M}\left(\frac{R}{R - h}\right)^2. \tag{4}$$

Now $M = \frac{4}{3}\pi R^3\gamma$; $M - M'' = \frac{4}{3}\pi\gamma_0 [R^3 - (R - h)^3]$;

$$\therefore \quad \frac{M''}{M} = 1 - \frac{\gamma_0}{\gamma}\left[1 - \left(\frac{R-h}{R}\right)^3\right] = \left(1 - \frac{\gamma_0}{\gamma}\right) + \frac{\gamma_0}{\gamma}\left(\frac{R-h}{R}\right)^3, \tag{5}$$

which substituted in equation (1), gives

$$\frac{F''}{F} = \left(1 - \frac{\gamma_0}{\gamma}\right)\left(\frac{R}{R-h}\right)^2 + \frac{\gamma_0}{\gamma}\left(\frac{R-h}{R}\right). \tag{6}$$

If h is a small fraction of R, equation (6) may be reduced to the approximate formula,

$$\frac{F''}{F} = 1 + \left(2 - 3\frac{\gamma_0}{\gamma}\right)\frac{h}{R}. \tag{7}$$

Corollary. — If the mean density of the outer layer of the Earth is less than two-thirds the mean density of the whole Earth, the weight of a body increases as it is taken below the surface of the Earth. (See Ex. 1, Art. 183.)

The mean density of the Earth being taken as 5.52 and that of the layer near the surface as 2.76, about the density of the rocks, makes $\gamma_0/\gamma = \frac{1}{2}$ and equation (7),

$$F''/F = W''/W = g''/g = 1 + h/2R. \tag{8}$$

That the weight of a body increases as it is taken below the surface has been shown by actual trial. From (8), the depth to which a body must be taken in order that it gain $\frac{1}{100}$ of one per cent in weight is approximately,

$$h = \frac{2 \times 20{,}902{,}000}{10{,}000} = 4180 \text{ ft.};$$

that is, a mass weighing a pound at the surface will weigh 1.0001 lb. at a depth of 4180 ft. below the surface.

Compared with case I, it may be seen that, under the conditions, for the same value of h, the gain in weight is one-fourth as much as the loss in weight when the body is above the surface, the same ratio of change applying to the value of g also.

191. Value of the Constant of Gravitation.* — From the foregoing as to the attraction of a sphere, it follows that the formula for the attraction of two particles,

$$F = K\frac{mm'}{r^2} \qquad [(1) \text{ Art. } 190]$$

will apply to two spheres, which are either homogeneous throughout or composed of a series of concentric shells, each one of which is of uniform density, m and m' being the masses of the spheres and r the distance between their centers.

By measuring the force of attraction between two spheres of known mass and distance apart, the value of K the constant of gravitation has been found. As stated in Art. 190, its numerical value will depend on the units used for the other quantities in the equation. The relation between the constant K and the mass of the Earth, taking the Earth as a sphere whose density is a function of the distance from the center may be shown as follows.

Let the units be the British gravitation units, and let R be the radius of the Earth in feet, M its mass, γ its mean density. Consider the attraction of the Earth on a body of mass m at the surface. By the formula (1) of Art. 190, the value of the attraction is KMm/R^2; but (since the unit force is the weight of a pound mass) expressed in pounds force, its measure is m. Hence, $m = KMm/R^2$ or

$$KM = R^2. \tag{1}$$

Since the value of R is known, either K or M can be found when the other is known. Putting for M its value in terms of γ,

$$K \cdot \tfrac{4}{3}\pi R^3\gamma = R^2 \quad \text{or} \quad K\gamma = \frac{3}{4\pi R}. \tag{2}$$

* Arts. 191 and 192 are based on Articles in Hoskins's *Theoretical Mechanics*.

Taking $\gamma = 345$ lbs. per cu. ft. and $R = 20{,}900{,}000$ ft. gives

$$K = \frac{3}{4\pi R\gamma} = 3/(4\pi \times 20{,}900{,}000 \times 345) = 3.31 \times 10^{-11}.$$

Otherwise, if the value of K, found by direct measurement of the attraction of two spherical bodies, is substituted in (2) the value of γ, the mean density of the Earth is found to be 5.527. The density of water being unity, and its weight 62.4 lbs. per cu. ft., the mean density of the Earth is about 345 lbs. per cu. ft., as used above.

192. Value of the Gravitation Unit of Mass.* — As stated in Art. 190, the force with which two particles of unit mass at a unit distance apart attract each other is equal to K, the constant of gravitation; this is evident from the equation,

$$F = K\frac{mm'}{r^2}. \quad [(1) \text{ Art. } 190.]$$

Let m pounds be the mass of each of two particles which, when one foot apart, attract each other with one pound force. Substituting $K = 3.31 \times 10^{-11}$, as given above, putting $F = 1$, $m = m'$, and $r = 1$; gives

$$m = 1/\sqrt{K} = 173{,}800 \text{ lb.}$$

If a mass equal to 173,800 pounds be taken as the unit mass, the constant K becomes unity and the formula for attraction is then

$$F = \frac{mm'}{r^2}.$$

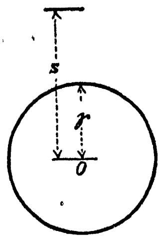
The *gravitation unit of mass* is thus shown to be a mass equal to about 173,800 pounds, distance being in feet and force in pounds-force.

193. Vertical Motion under the Attraction of the Earth. —Let the Earth be taken as in Example 3, Art. 190, r as the radius and s the distance of the moving particle from

* See Footnote on page 378.

the center O. Taking distance, velocity, and acceleration as positive outward, then, as in (1) Ex. 3,

$$\frac{F'}{F} = \frac{g'}{g} = \frac{r^2}{s^2}, \quad \text{or} \quad g' = \frac{gr^2}{s^2}.$$



Since $\dfrac{dv}{dt} = a$ and $\dfrac{ds}{dt} = v$, eliminating dt gives $v\,dv = a\,ds$.

Hence, $a = -g' = -\dfrac{gr^2}{s^2}$, neglecting air resistance, which gives

$$\int_0^v v\,dv = \int_s^r -gr^2 s^{-2}\,ds;$$

integrating, $$\frac{v^2}{2} = gr^2 s^{-1}\Big]_s^r = gr^2\left(\frac{1}{r} - \frac{1}{s}\right);$$

then, $$v^2 = 2\,gr^2\left(\frac{1}{r} - \frac{1}{s}\right) \tag{1}$$

gives the velocity of a particle towards the Earth from any distance s.

For the velocity acquired by a body in falling to the surface from a height h, put $s = r + h$ in (1), giving,

$$v^2 = 2\,gr^2\left(\frac{1}{r} - \frac{1}{r+h}\right) \tag{1'}$$

$$= 2\,gh\left(\frac{r}{r+h}\right) \tag{2}$$

$$= 2\,gh, \text{ approximately,}$$

if h is small, which is the formula when g is constant, as it is taken near the surface. When $\dfrac{h}{r}$ is small, putting

$$\frac{r}{r+h} = \left(1 + \frac{h}{r}\right)^{-1} = 1 - \left(\frac{h}{r}\right) + \left(\frac{h}{r}\right)^2 - \left(\frac{h}{r}\right)^3 + \cdots,$$

(2) becomes

$$v^2 = 2\,gh\,[1 - (h/r) + (h/r)^2 - (h/r)^3 + \cdots]. \tag{3}$$

By taking any number of terms of this series, an approximate result may be gotten as nearly correct as desired. If, in (1), $s = \infty$, $v = \sqrt{2\,gr}$; so if a body fell toward the Earth from an infinite distance, its velocity, neglecting air resistance, would be $\sqrt{2\,gr} = 6.95$ miles per second, for $r = 3960$ miles. If falling from a finite distance s, the velocity must be less than this. Hence, a body can never reach the earth with this velocity; and if air resistance is considered, the velocity for $s = \infty$ is less than $\sqrt{2\,gr}$. If projected outward with velocity $\sqrt{2\,gr}$ and air resistance be neglected, the body would go an infinite distance. This velocity is called the *critical velocity* or velocity of escape, for under the conditions it is supposed that certain particles of the atmosphere may escape from the attraction of the Earth.

In this connection, it is to be recalled that due to the Earth's rotation, there is at its surface a centrifugal force $mg/289$, exerted by a particle of mass m, which lessens the value that g would otherwise have.

194.* Necessary Limit to the Height of the Atmosphere. — The centrifugal force of a particle of mass m on the surface of the Earth is $m\omega^2 r = \dfrac{mg}{289}$, and at a distance s from the center it would be $m\omega^2 s = \dfrac{mgs}{289\,r}$. The Earth's attraction at that distance being $\dfrac{mgr^2}{s^2}$, in order that the particle be retained in its path these two forces must equal each other;

$$\therefore \quad \frac{mgs}{289\,r} = \frac{mgr^2}{s^2},$$

or
$$s^3 = 289\,r^3,$$

hence
$$s = \sqrt[3]{289}\,r = 6.6\,r$$
$$= 26{,}000 \text{ miles approximately};$$

that is, a height above the surface of about 22,000 miles. The actual height of the atmosphere is probably much less than

* Bowser's *Hydromechanics*.

this. The estimates of the height by various scientists have been very divergent — from 40 miles to 216 miles; but the latter appears to be the most likely, for meteors have been observed at an altitude of more than 200 miles and, as they become luminous only when they are heated by contact with the air, this is evidence that some atmosphere exists at that height. It is supposed that at a height much less than 5.6 r, the air may be liquefied by extreme cold.

195.* Motion in Resisting Medium. — Consider the motion of a body near the surface of the Earth under the action of gravity taken as a constant force and the air taken as a resisting medium of uniform density, the resistance varying as the square of the velocity.

Let a particle be supposed to descend towards the Earth from rest, and let s be the distance of the particle from the starting point at any time t, gk^2 the resistance of the air on a particle for a unit of velocity — gk^2 being the *coefficient of resistance*. The resistance of the air at the distance s from the origin will be $gk^2\left(\frac{ds}{dt}\right)^2$, acting upwards, while g acts downwards, the mass being a unit.

The equation of motion is

$$\frac{d^2s}{dt^2} = g - gk^2\left(\frac{ds}{dt}\right)^2, \tag{1}$$

or

$$g\,dt = \frac{d\left(\frac{ds}{dt}\right)}{1 - k^2\left(\frac{ds}{dt}\right)^2}.$$

Integrating,

$$gt = \frac{1}{2k}\log\frac{1 + k\frac{ds}{dt}}{1 - k\frac{ds}{dt}}.$$

$t = 0$, $v = 0$, giving $C = 0$.

* Bowser's *Analytic Mechanics.*

Passing to exponentials,

$$\frac{ds}{dt} = \frac{1}{k} \frac{e^{kgt} - e^{-kgt}}{e^{kgt} + e^{-kgt}}, \tag{2}$$

which gives the velocity in terms of the time. To get it in terms of the space, from (1),

$$\frac{k^2 d\left(\frac{ds}{dt}\right)^2}{1 - k^2\left(\frac{ds}{dt}\right)^2} = 2\,gk^2 ds;$$

$$\therefore \quad \log\left[1 - k^2\left(\frac{ds}{dt}\right)^2\right] = -2\,gk^2 s,\ s = 0,\ v = 0;\ C_1 = 0, \tag{3}$$

or

$$\left(\frac{ds}{dt}\right)^2 = \frac{1}{k^2}(1 - e^{-2gk^2 s}), \tag{4}$$

which gives the velocity in terms of the distance. Also, integrating (2);

$$gk^2 s = \log\left(e^{kgt} + e^{-kgt}\right) - \log 2;$$

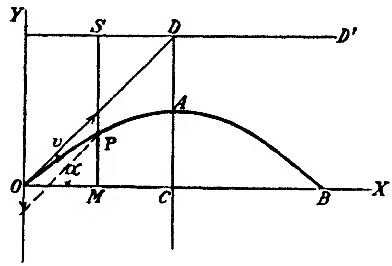
$$\therefore \quad 2\,e^{gk^2 s} = e^{kgt} + e^{-kgt}, \tag{5}$$

which gives the relation between the distance and the time of falling through it.

As the time increases the term e^{-kgt} diminishes and from (5) the space increases, becoming infinite when the time is infinite; but from (2) as the time increases the velocity becomes more nearly uniform, and when $t = \infty$, the velocity $= 1/k$; and although this state is never reached, yet it is that to which the motion approaches.

196. Motion of a Projectile. — If a body be projected with a given velocity in a direction not vertical and be acted on by gravity only, neglecting the resistance of the air, it is called a *projectile*. The path, called the *trajectory*, will result from a combination of the motions due to the velocity of projection and to g, the vertical acceleration of gravity. Let the plane in which a particle is projected with a velocity

v be the plane of XY, and let the line of projection be inclined at an angle α to the x-axis, making $v \cos \alpha$ and $v \sin \alpha$ the resolved parts of the velocity of projection along the axes of x and y.



Let (x, y) be the position of the particle P at the time t; then, since the horizontal acceleration is zero and the vertical acceleration negative,

$$\frac{d^2x}{dt^2} = 0; \quad \frac{d^2y}{dt^2} = -g.$$

Taking the first and second integrals of these equations, determining the value of the constants of integration corresponding to $t = 0$ and $t = t$, gives

$$\frac{dx}{dt} = v \cos \alpha; \quad \frac{dy}{dt} = v \sin \alpha - gt; \tag{1}$$

$$x = v \cos \alpha t; \quad y = v \sin \alpha t - \tfrac{1}{2} gt^2. \tag{2}$$

Equations (1) and (2) give the coördinates of the particle and its velocity parallel to either axis at any time t.

Eliminating t between equations (2), gives

$$y = x \tan \alpha - \frac{gx^2}{2 v^2 \cos^2 \alpha}, \tag{3}$$

which is the equation of the trajectory, and shows that the path of the particle is a parabola.

Putting equation (3) in the form

$$x^2 - \frac{2 v^2 \sin \alpha \cos \alpha}{g} x = - \frac{2 v^2 \cos^2 \alpha}{g} y,$$

or

$$\left(x - \frac{v^2 \sin\alpha \cos\alpha}{g}\right)^2 = -\frac{2v^2\cos^2\alpha}{g}\left(y - \frac{v^2\sin^2\alpha}{2g}\right), \tag{4}$$

and comparing this with the equation of a parabola,

$$(x - h)^2 = -2p(y - k),$$

it is seen that:

$$\text{the abscissa of the vertex} = \frac{v^2 \sin\alpha\cos\alpha}{g}; \tag{5}$$

$$\text{the ordinate of the vertex} = \frac{v^2\sin^2\alpha}{2g}; \tag{6}$$

$$\text{the latus rectum} = -\frac{2v^2\cos^2\alpha}{g}. \tag{7}$$

By transferring the origin to the vertex, (4) becomes

$$x^2 = -\frac{2v^2\cos^2\alpha}{g}y, \tag{8}$$

which is the equation of a parabola with its axis vertical and the vertex the highest point of the curve.

The distance between the point of projection and the point where the projectile strikes the horizontal plane, called the *Range*, is

$$OB = x = \frac{v^2 \sin 2\alpha}{g}, \tag{9}$$

when $y = 0$, from (3), which is evident geometrically, since $OB = 2\,OC$; that is, the range is equal to twice the abscissa of the vertex.

It follows from (9) that the range is greatest for a given velocity of projection, when $\alpha = 45°$, in which case the range $= \dfrac{v^2}{g}$. It appears from (9) that the range is the same for the complement of α as for α. The greatest height CA is given by (6) which, when $\alpha = 45°$, becomes $v^2/4g$.

The height of the directrix,

$$CD = CA + AD = \frac{v^2\sin^2\alpha}{2g} + \frac{1}{4}\frac{2v^2\cos^2\alpha}{g} = \frac{v^2}{2g}.$$

Hence, when $\alpha = 45°$ the focus of the parabola is in the horizontal line through the point of projection, for then $CA = \frac{1}{2} CD$.

To find the velocity V at any point of the path, from (1),

$$V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$
$$= v^2 \cos^2 \alpha + (v^2 \sin^2 \alpha - 2v \sin \alpha gt + g^2t^2)$$
$$= v^2 - 2gy, \quad \text{or} \quad \frac{V^2}{2g} = \frac{v^2}{2g} - y = MS - MP = PS.$$

Since $\frac{V^2}{2g}$ is the height through which a particle must fall from rest to acquire a velocity V, it follows that the velocity at any point P on the curve is that which the particle falling freely through the vertical height SP would acquire; that is, in falling from the directrix to the curve; and the velocity of projection at O is that which the particle would acquire in falling freely through the height CD.

For the time of flight, put $y = 0$ in (3) and solve for $x = \frac{2v^2 \sin \alpha \cos \alpha}{g}$, which divided by $v \cos \alpha$ gives, time of flight $= \frac{2v \sin \alpha}{g}$; or in (2) put $y = 0$ and solve for t, giving $t = 0$ and $t = \frac{2v \sin \alpha}{g}$, as before.

197. Motion of Projectile in Resisting Medium. — If the resistance of the air is taken to vary as the square of the velocity and the angle of projection is very small, the projectile rising but a very little above the horizontal, the equation of the trajectory above the horizontal line can be found. Thus the equation

$$y = x \tan \alpha - \frac{gx^2}{2v^2 \cos^2 \alpha} - \frac{gkx^3}{3v^2 \cos^2 \alpha} - \cdots,$$

may be derived under such conditions; where the first two

terms represent the trajectory neglecting air resistance, as found in (3), Art. 196.

For the high velocities of cannon-balls the trajectory is found to be very different from the parabolic path and the range much less than that deduced for it.

Experiments show that the angle of projection for greatest range is about 34°, rather than 45°, as deduced for the parabolic path.

The simplest formula for making out a range table is Hélie's:

$$y = x \tan \alpha - \frac{gx^2}{2 \cos^2 \alpha}\left(\frac{1}{v_0{}^2} + \frac{kx}{v_0}\right),$$

where $k = 0.0000000458 \dfrac{d^2}{w}$, d being the diameter of the projectile in inches, and w its weight in pounds.

In addition to the resistance of the air, allowance has to be made in firing for the *drift*, that is, the tendency for most projectiles to bear to the right upon leaving the gun, due to the right-handed rotation given to the projectile.

198. Newton's Verification. — The greatest of Newton's achievements is considered an achievement of the *imagination*, his conception of the universality of natural law. At an early age he (in 1666) conceived with his far-reaching mind the then daring idea that the sublime, inscrutable, central force was nothing but commonplace gravity, known to exist on and near the earth. He verified his idea first in the case of the moon. He discovered that the same acceleration that controls the motion of an object near the earth also prevented the moon from moving away in a rectilinear path from the earth, and that its tangential velocity prevented it from falling to the earth.

Owing to an inaccurate value of the earth's radius which was in use at the time Newton first made the computation,* the result then obtained seemed to show that the law of attraction was not that of inverse squares.

* See Illustrative Example, page 100.

Records show that Newton, although unshaken in his belief, laid aside his calculations; and it was not until thirteen years afterwards that, a new determination of the radius having been made, he repeated the investigation and found the verification sought for.

Five years later (in 1684) he was induced to consider the whole subject of gravitation; and then he solved the supplementary problems in regard to the attraction of a sphere for an external particle, which established his theory — now known as *Newton's Law of Universal Gravitation.*

226.* The Need and Fruitfulness of the Solution of Differential Equations. — Attention has heretofore been called to the need of finding the inverse of a rate, in solving many problems that arise in everyday life as well as in science. In fact the inverse problem is more often the real question demanding solution. It has been shown (Ex. 5, Art. 115) how, when the acceleration, the rate of change of the speed of a moving body, is known, the velocity and the distance for any time are found by the solving of a differential equation. (Ex. 1, Art. 173.)

It has been shown (Art. 42), that when a function has the general form $y = ae^{bx}$, the rate of change is proportional to the function itself, and that so many changes in Nature occur in this way that the law of change, known as the *Compound Interest Law,* is also called the *Law of Organic Growth.* Now, if it is known that some function changes at a rate proportional to itself, expressing this by the differential equation,

$$\frac{dy}{dx} = ky, \quad \text{or} \quad k\,dx = \frac{dy}{y};$$

then, $$kx = \int \frac{dy}{y} = \log_e y + c, \quad \text{or} \quad y = e^{kx-c} = Ce^{kx},$$

where $C = e^{-c}$ is an arbitrary constant.

* Applied Calculus.

The only function whose rate of change is proportional to itself is thus shown to be of the form Ce^{kx} (or ae^{bx}), where C and k (or a and b) are arbitrary, and k (or b) is the factor of proportionality. This may be expressed also by the statement, that the only function, whose relative rate of change (logarithmic derivative) is constant, is Ce^{kx} (or ae^{bx}).

It has been shown (Art. 73), that when a point has *simple harmonic motion* its relative acceleration is a negative constant. Thus, when the displacement of a point is given by the equation $y = a \sin (\omega t - \alpha)$, there results the differential equation $\frac{d^2y}{dt^2} = -\omega^2 y$, where ω is constant, and hence the relative acceleration is $\frac{d^2y}{dt^2} \Big/ y = -\omega^2$.

Conversely, when the motion, as in a vibration, is due to a force that increases with the distance from the central position, the acceleration, being according to Newton's second law of motion proportional to the force, is

$$a_t = \frac{d^2s}{dt^2} = -k^2 s,$$

where, as the force acts towards the origin, the acceleration is negative when s is positive and positive when s is negative. From the relation $v\,dv = a_t\,ds$, gotten by eliminating dt in $dv/dt = a_t$ and $ds/dt = v$;

$$\int v\,dv = \int -k^2 s\,ds; \quad \therefore \quad v^2 = C_1 - k^2 s^2;$$

putting $C_1 = k^2 a^2$,

$$v = \frac{ds}{dt} = k\sqrt{a^2 - s^2}; \quad \int \frac{ds}{\sqrt{a^2 - s^2}} = \int k\,dt;$$

whence $$\sin^{-1}\left(\frac{s}{a}\right) = kt + C_2$$

or $$s = a \sin (kt + C_2) = A \sin kt + B \cos kt,$$

where $A = a \cos C_2$ and $B = a \sin C_2$ are arbitrary constants. This equation for s is the characteristic equation of *simple harmonic motion;* the amplitude of the motion is a, the period is $2\pi/k$, and the phase is $-C_2/k$.

Thus, it is found that, when the acceleration along a straight line is a negative constant times the distance from a fixed point, the only motion resulting is the *simple harmonic motion.*

In general, it has been shown that, whenever the rate of change of a function of a single independent variable is known and also the value of the function for some one value of the variable, it is possible to find by integration the value of the function for any value of the variable.

Hence it has followed that the solution of a differential equation gives the area under any curve whose equation is known, thus solving the problem that had baffled the mathematicians of the ages before the discovery of this general method of effecting the quadrature of curves of any degree.

When it is recalled that the magnitude of any quantity whatever, whether of volume, mass, weight, force, work, etc., may be represented by an area under a curve, the fruitfulness of the solution of many differential equations is recognized.

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